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ELEMENTS
OF
PLANE AND SPHERICAL
TRIGONOMETRY,

WRITTEN ORIGINALLY IN RUSSIAN,

AND

TRANSLATED INTO ENGLISH BY THE AUTHORS,

BASIL NIKITIN, M. A. Inspector,

AND

PROCHOR SOUVOROFF, M. A. Vice-inspector,

OF THE

IMPERIAL ACADEMY OF CADETS AT CRONSTADT.

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PREFACE.

P R E F A C E.

THE authors take the liberty to lay this ſmall work before the *Britiſh* nation, as the fruit of that learning, which was ſo cordially and ſo liberally communicated to them in *England*, and as a ſmall token of gratitude for the great honours, which they, the firſt of their nation, have been kindly thought worthy to receive there.

A work of this kind cannot indeed be offered before better and more proper judges; not only on account of the very great knowledge of the *Britiſh* nation in theſe, as well as in other matters, but alſo becauſe the *Britiſh* mathematicians have been the greateſt, nay, we will ſay, the only improvers of Trigonometry within theſe two centuries. This is ſo very true, that not to mention the extraordinary inventions of Lord *Neper* in this part of mathematics, nor the analogies, wherein ſines and tangents of half-arches are uſed, nor the applications of Trigonometry to ſtereographic projection of the ſphere, for all which we are indebted to them; the very words of *Trigonometry*, *coſine*, *cotangent*, &c. have been firſt uſed by the writers of that nation.

The moſt conſpicuous authors of both Trigonometries amongſt the *Britiſh*, who have been conſulted by us, are *Caſwell*, in *Wallis*'s Works, *Keil*, *Simpſon*, *Robertſon*, Mr. *Thomas Simpſon*, *Emerſon*, and *Martin*; and of thoſe who have treated of them occaſionally, *Oughtred*, in his Clavis Mathematica, and Circles of Proportion, *Wallis*, *Jones*, *Wingate*, *Sherwin*, and *Gardiner*, in their Tables of Logarithms; Sir *Iſaac Newton*, in his Univ. Arithm. Geometrical Problem II. *Harris* and *Chambers* in their dictionaries. Plane Trigonometry alone has been treated by *Ronayne*, Mr. *Thomas Simpſon*, *Maſeres*, and *Muller*. However, the merits of ſome foreigners alſo cannot, without injuſtice, be ſuppreſſed. Such are *Copernicus*, in his Aſtronomia Inſtaurata; *Balanus*, a modern *Greek*;

Simon Stevin, commented upon by *Albert Girard*; *Clavius*; *M. de la Caille*; *M. de la Lande*, in his Aſtronomie, tom. III. *de Chales*; *Ozanam*; *Segnerus*; and the labours of *Schottus*, *Tacquet*, and others, are commendable. We need not mention the parents of theſe ſciences, *Theodoſius*, *Ptolemy*, *Menelaus*, and *Regiomontanus*.

The *Engliſh* nation may boaſt of a man, who alone invented and perfected two branches of the mathematics, and made innumerable, and almoſt ſupernatural improvements in others. We mean Sir *Iſaac Newton*, in his optics and fluxions; but in general, the genius of mankind opens itſelf, and ſtrengthens, and improves, but ſlowly; and the joined labours of many men and many ages are requiſite to complete and perfect one ſingle ſcience. This lot ſeems to have been incident, amongſt other innumerable arts and ſciences, to the two Trigonometries. For notwithſtanding the labours and exertions of ſo many eminent men, it will appear to every curious and impartial man, that in this branch of mathematics many things are deficient, many ſuperfluous; ſome are too general, others too particular; ſome are too much dwelt upon, others want a great deal of explanation; in many there is hardly any order, or connexion, or demonſtration, in ſome too much unneceſſary preciſion. For what elſe may be the reaſon of doubts ariſing in ſolutions both of plane and ſpherical triangles, but the want of accurate determinations and explanations? From what have proceeded diſputes in Spherical Trigonometry, not ſolved either by *Cunn*, or *Ham*, but from the inaccurate notion of a ſupplemental triangle? One may appeal to every attentive and competent reader, whether he does not fall into many doubts and perplexities throughout the whole Spherical Trigonometry: and to what can this be owing, but to the want of ſufficient principles, the neglect of enumerating and diſtinguiſhing caſes of a propoſition, and the inattention to rendering the ſubject as complete as poſſible? It may be pardonable, in the opinion of the true judges of theſe matters, to advance, that there is as yet no claſſical book of Trigonometry in any language, ſuch as is fit to give to learners a ſolid foundation in them, and ſuch as are *Euclid*'s Elements, *Archimedes* de Sphaera et Cylindro, and Dr. *Hamilton*'s Conic Sections.

The authors mean to make good, what they have advanced; and this, they think, they can do no better than by ſhewing, what they have performed in their own work, and with what views and reaſons; which, they hope, may at the ſame time ſerve as an apology for their engaging in this undertaking, after the pens of ſo many eminent men, who have highly deſerved of theſe ſciences and of mankind.

I. In the firſt place, care was taken to divide the work into a ſeries of ſections, prefixing the ſubject of each. This ſeemed to keep the whole more compact, and to prevent things belonging to the ſame ſubject from lying broken off and detached. It is for this reaſon, that the lemma in Book I. and prop. 3. and 7. Book II. as well as the circular elements, ſeemed diſpoſed in their proper places; which is not the caſe in other writers. This alſo probably may ſerve towards a more diſtinct comprehenſion and view of the whole, and conſequently towards the eaſier imprinting of it in the memory. If in this reſpect corr. 1 & 2. to def. 4. book II. be objected againſt; their place may perhaps be defended from their vicinity to their proper ſubject; and it may be mentioned, once for all, that, leſt the number of propoſitions ſhould be too much increaſed, and the volume ſwelled, the authors thought themſelves obliged, in conformity with the prevailing taſte for mathematics, and in imitation of the modern writers upon ſuch ſubjects, to throw ſeveral things into corollaries, which might very well have made propoſitions of themſelves.

II. Throughout the whole work, it was greatly ſtudied, that the ſubject ſhould become as general, as full and complete, and as accurate and preciſe as poſſible.

In the very beginning it ſeemed neceſſary to introduce the ſubject of Trigonometries, and as it were to connect it with thoſe of Geometry and Arithmetic. This is done in def. 1. and 2. The firſt part of def. 1. makes a lemma by itſelf (viz. 1ſt.) in Dr. *Simpſon*, and is *Keil*'s *Euclid*, 2. cor. 33. 6. There alſo it is ſhewn, how it comes, that, in Trigonometry, arches and angles are conſidered to be almoſt of the ſame import.

In Book I. def. 3. it ſeemed neceſſary to accommodate the notion of ſupplements and complements, not only to arches leſſer than a ſemicircumference, but to thoſe alſo, which are greater than it. This, at the ſame time that it completes the ſubject of this part of Trigonometry, by conſidering all ſorts of arches of a circle, is not without its actual uſe, not in ſpherics only, but alſo in other parts of Mathematics. But it is ſurprizing, how here the few authors, that treat of this matter, diſagree with one another, nay diſagree with themſelves. For which reaſon def. 3. has undergone a mature conſideration, and been inſtanced in all. thoſe caſes, where it tacitly enters, e. g. in Book I. def. 4. cor. 1; def. 8. cor. 2; def. 9. cor. 1, 2, & 6; prop. 2. & cor.; prop. 6.; and in Book II. lemm. 1, 2, 3, 4. It was moreover thought proper to add a ſcholium at the end of prop. 4. Book I. in order to ſhew, how the trigonometrical lines, belonging to all ſorts of arches of a circle, increaſe and decreaſe; that thus the ſubject might be fully explained and diſcuſſed. Arches, however, greater than the circumference of a whole circle, have been forborn, on account of their little uſe; though their ſupplements and complements, and the trigonometrical lines, appertaining to them, in all the circles after the firſt, are to be underſtood exactly the ſame way as in that firſt circle, adding only ſo many circumferences. And here, by the by, it may be obſerved, that the reaſon, why in Algebra and Fluxions, expreſſions for trigonometrical lines always run out into infinite ſerieſes, is becauſe the number of arches, to which any one of ſuch lines belongs is always infinite.

Several other definitions, together with their corollaries, had ſome pains beſtowed upon them; though of them we ſhall afterwards have occaſion to ſpeak ſummarily.

Prop. 5. has ſuch a demonſtration, or rather explanation, as the ſubject admits of. As this prop. is only nearly true, and not ſtrictly ſo; and is indiſpenſable for the conſtruction of the canon in the way propoſed in the book; it has been explained and enlarged upon as much as poſſible; and for this, a ſtep has been uſed, which is entirely new.

In prop. 6. it was not without ſome trouble, that its original univerſality (as it ſtands in its Author, Marquis *de l'Hoſpital*, Con. Sect. book 10. lem. 1. §. 442) was reduced to trigonometrical purpoſes, and the prop. properly limited and extended to chords of arches as well as to chords of their ſupplements. Its 2d. cor. has a ſimilar limitation and application; and the 4th. cor. is entirely upon a new footing, both in its enunciation and demonſtration. The uſe of this prop. may be extended to ſubjects not merely trigonometrical: ſince, by means of prop. 7. which is dependent upon it, the ratio of the diameter of a circle to its circumference may be ſought, as is done by Dr. *Robertſon* in his Navigation. For the chord $E^{xi}B$ (See Pl. 1. fig. 5.) is the ſide of an equilateral and equiangular polygon of 12288 ſides inſcribed into a circle; whence the whole perimeter of that polygon is known. By an obvious analogy one may find the perimeter of a ſimilar polygon circumſcribed about that circle; and the arithmetical mean between theſe too perimeters, as far as it agrees with each of them, is the circumference of that circle. Thus in the circle, whoſe diameter is 2, the circumference is 6.2831853; therefore the diameter of a circle is to its circumference, as 2 to 6.2831853, or as 1 to 3.1415926; and by approximation of ratios, as 113 to 355, 7 to 22, 1 to 3. It might be obſerved here by the way, that the circumference is much nearer to the firſt of two arithmetical means between the perimeters of the inſcribed and circumſcribed polygons before mentioned; both that mean and the circumference being 6.28318,53071.7958. But let us not digreſs.

In the lemma, the former incomplete demonſtrations have been let alone; and a new one (it is hoped, much preferable to them) is ſubſtituted, in the manner of thoſe of firſt prop. of Elem. Book II.

Prop. 11. with its cor. has been introduced, on account of the ſhortneſs of ſolution, which they afford, eſpecially as ſuch a ſolution is much uſed in ſome aſtronomical calculations.

To prop. 13. the 1ſt. cor. is added for ſimilar reaſons; and in reſpect to cor. 2. it completes the ſubject of the prop. by finding the baſe alſo; and beſides the propoſed way to find it, is ſhorter than the common one, ſince ſome of the ſteps neceſſary for it, will be found to be previouſly calculated.

Prop. 14. and its corollaries deſerve a particular examination. It is hard to ſay, by whom they were invented, though moſt probably by the *Engliſh*; and perhaps corr. 3 & 2. of prop. 29. in ſpherics, have given riſe to them all, as they are to be found in moſt books of the laſt age. They are all to be ſeen in *Caſwell* apud *Wallace*, *Newton* Univ. Arithm. Geom. Probl. 11. *Thos. Simpſon*'s Algebra, Geom. Probl. 15. Dr. *Robertſon*'s Navigation, *Emerſon*, and *Martin*. The analogy of the prop. in particular, is to be met with in *Trigonometria Britannica*, *Sherwin*'s Tables, *de la Caille*, Dr. *Simpſon*, and *Ward*. Notwithſtanding M. *de la Caille*'s diſpleaſure againſt it, it ſeems by far preferable to all the other analogies, there being no change of its terms, except in the ranging of them, when more than one angle is to be found; which is a great advantage, and particularly in a correſponding caſe in ſpherics. And to this purpoſe, we have invented a cor. (the 4th.), which ſeems a great improvement upon the prop. itſelf. However, the other analogies are ſubjoined alſo, not only that the ſubject may be complete, but that there may be a choice of them; eſpecially as thoſe in cor. 3. are neceſſary for ſolution by the logarithmical ſcale. With regard to demonſtration, the old and perplexed one (uſed formerly for the area of a triangle, and accommodated here by Dr. *Simpſon*) is laid aſide, and another and an eaſier demonſtration is ſubſtituted inſtead of it, after the manner of Dr. *Robertſon*'s, but greatly improved by the change of a ſide of the triangle; the ſame indeed nearly, which has been communicated in *Ruſſia* ſome years ago by Mr. Profeſſor *Robiſon* of *Glaſgow*. A ſimilar demonſtration may be ſeen in an inconſiderable *Ruſſian* book, upon the coſine of half the angle; though it is uncertain, whence it is taken. There are four other analogies relating to the ſame ſubject, viz. upon the ſine, coſine, tangent, and cotangent of the whole angle; but theſe being rather perplexed, and not ſo eaſily manageable, and beſides affording a doubtful ſolution, which therefore muſt be determined from other principles, it was thought proper to paſs them by.— The foregoing conſiderations may all be applied to prop. 29 & 30. of ſpherics and their corollaries, with this difference only, that leſs aſſiſtance was to be had there. It is hoped, the authors will be pardoned for introducing here the 5th. cor. belonging, in ſtrictneſs, to geometry: its very great importance, and the ſuperior and extraordinary ſimplicity of its demonſtration (which is entirely new), as well as ſome ſmall relation to the ſubject, may poſſibly, juſtify its admittance.

Prop. 15. is worded in a new and complete way; and the caſe of ſolutions, dependant upon it, is made incomparably more accurate, and more general and ſatisfactory.

But in Book II. i. e. in Spherical Trigonometry, the greateſt pains were to be taken, and the greateſt difficulties to be overcome. For though it is not to be denied, that Spherical Trigonometry is not ſo eaſy as the Plane, as it wants thoſe previous helps and that determination, which the Plane Trigonometry has; yet, when out of three parts of a propoſition one only or two are laid down; when a propoſition is demonſtrated only in one caſe out of ſeveral; when, on account of bad definitions, ſeveral things, wanting demonſtration, are paſſed by, or dignified with the name of axioms; when an argument turns in circulo, as it is called; when whole ſteps are omitted; when, inſtead of a direct way, we go round about; when things are ſcattered about without order; when a whole ſet of triangles is neglected, &c. &c. ſurely, all this is not the fault of Spherical Trigonometry.

In the very outſet it is a matter of ſurprize, how ſuch important, though eaſy, things, as circles of the ſphere, and ſpherical right and obtuſe angles have been left to this day undefined; which defect is therefore ſupplied here in def. 1 & 7. Definitions of ſuch angles are the more neceſſary, as the inclination of planes is, with *Euclid*, acute angled only; obtuſe he does not mention, and a right one he calls by another name. *Menelaus* alone has defined a ſpherical right angle, and very probably with the ſame view as we have done.

The writers of ſpherics define for us here the axis of the ſphere, which is one, and invariable in aſtronomy only, and of which there is an indefinite number in ſpherics; on the contrary, of the axis of a circle of the ſphere, which is ſo neceſſary here, almoſt every body is ſilent.

The whole doctrine of axes and poles is to this day both incomplete and inaccurate. The authors endeavoured to their utmoſt, to remedy ſuch extraordinary defects in ſo important a ſubject, in def. 3. corr; prop. 1. corr.; and prop. 2. and corr. which, they preſume, will complete the ſubject effectually, ſince they are made according to all poſſible combinations, from which any thing may follow. The authors think themſelves particularly lucky in having hit upon ſuch definitions, as 2 & 3. adapted, of all authors, by M. *de la Caille* only; though he did not imagine, how much ſpherics might be enriched by means of them. For the conſequences obviouſly following from them are ſurpriſing; nor leſs ſo is the eaſineſs with which they are derived, they being evident from axioms alone, or at moſt from principles nearly as manifeſt. In truth, the ſubject of poles of circles, as it is laid down here, ſeems exhauſted: ſeveral of their properties are exhibited, over and above thoſe in *Theodoſius*'s Spherics, and Dr. *Barrow*'s additions thereto; and this is done in a leſſer number of lines, than they have pages.

Prop. 3. is made more general, and its demonſtration more accurate.

Cor. to prop. 4. ſeem to complete the ſubject of parallel circles, ſo neceſſary in Aſtronomy and Navigation.

In prop. 5. and cor. the confuſed and inaccurate ideas of *arches being meaſures of angles*, of *arches being equal to angles*, and of *arches being the ſupplements and complements of angles*, and *v. v.* ſo much prevailing even among the beſt geometricians, are attempted to be rectified: for it is manifeſt enough, that nothing can be a meaſure of another thing, or equal to it, or a ſupplement and complement of it, unleſs it be homogeneous with it. For want of ſuch a plain conſideration, and afterwards moſt probably from habit, people have debaſed many propoſitions, both in their enunciations and demonſtrations; and often it is not without ſome trouble that they are corrected. The propoſition before us is the chief amongſt them; it is indeed their fountainhead in ſpherics; thoſe derived from it are its cor; prop. 7; prop. 20—23; prop. 30. and its cor.

Prop. 6. has a new demonſtration; and cor. 2 & 3. are introduced, that they, along with 9 def. may ſerve for teſts of ſpherical triangles; which are neceſſary in prop. 11, 16, 17, 24, 25, &c. which *Segnerus* alſo ſeems to have had in view.

Prop. 7. claims particular notice on account of its uſe, of its application, and even on account of the diſputes it has occaſioned. Its application is ſo very extenſive, that the doctrine of ſpherics, by means only of it, may be reduced almoſt to half the number of its propoſitions. The invention of it may be aſcribed perhaps to *Philip Langſberg*. Vid. *Simon Stevin* Liv. 3. de la Coſmographie, prop. 31. & *Alb. Girard* in loc. It treats of what is called *ſupplemental* triangles; but ſo ſmall is the regard for preciſion in ſpheric writers, that out of full ſeven triangles, which the conſtruction of the propoſition may afford, it is not determined by any body, which triangle particularly is to be taken; and what ſeems even worſe, in the propoſition they uſe one triangle, and in the ſolutions another. On this account we have been obliged to form for it an enunciation entirely new; and were happy to find afterwards, that Mr. *Cotes* in his Æſtim. Error. lem. 4. gives the firſt part of it exactly the ſame. Its demonſtration alſo was to be corrected: nor without trouble, as the inaccuracy of *arches being ſupplements of angles*, and *v. v.* ſo much running through this propoſition, was to be rectified, and *that* from ſome expreſſions with negative ſigns prefixed to them. It is from the vagueneſs of the propoſition, before mentioned, and from miſunderſtanding the terms *ſupplement* and *complement*, that diſputes have ariſen in ſpherics: theſe may be ſeen at the end of *Cunn*'s *Euclid*, in his remarks and the appendix. Whatever be the miſtakes of Mr. *Heynes*, with whom we are not acquainted, the reſpectable names of Dr. *Keil*, Mr. *Caſwell*, and Dr. *Harris*, whom Mr. *Cunn* joins in company with *Heynes*, are treated by him certainly injuriouſly; eſpecially as he himſelf had not examined his ſubject with ſufficient attention. His own rule, recommended in preference to the common one of theirs, is indeed true, becauſe three of them are ſo, and his is among the number; but it is more troubleſome to the memory. Mr. *Ham*, who choſe himſelf an arbitrator between Dr. *Keil* and *Cunn*, holds the ſcales of juſtice equally, inclines to neither of the parties, commends both, but approves of neither, and at laſt awards his own rule, which, notwithſtanding, is much more unmanageable than either of theirs; it uſing ſubtraction of natural verſed ſines, to whoſe difference therefore (and every one knows the thing is not eaſy) logarithms are to be accommodated. But it were time, long ago, to bury theſe worthleſs diſputes in oblivion, that learners of ſpherics ſhould not be diſcouraged by ſeeing them printed and reprinted ſo often.

Prop. 8 & 9. are ſomewhat corrected after Dr. *Simpſon*. He ſeems to be the firſt that began to conſider the ſides of ſpherical triangles as planes, which conſtitute a ſolid angle at the center of the ſphere.

Of prop. 10. no one gives an accurate demonſtration, except *Menelaus*: Dr. *Keil*'s need not be mentioned, and Dr. *Simpſon*'s leaves out a caſe, and is at the ſame time very prolix. That which is offered here, is conformable to the demonſtrations of the two preceding propoſitions, ſeems remarkably ſhort and eaſy, and is derived from Dr. *Simpſon*'s Elements of *Euclid*, Book XI. prop. A. Nor let it be here objected, that that ſame propoſition of the Doctor's demonſtrates but one caſe out of ſeveral: the fact is ſo; he is indeed never enough to be praiſed for his merits in his edition of *Euclid*; but it muſt be confeſſed, that there are ſtill many inaccuracies, lurking not in the author's only, but alſo in the editor's propoſitions, and particularly in thoſe of the laſt two books. But as, in our edition of the elements in *Ruſſian*, we have given an accurate and complete demonſtration of that propoſition (wherein we make two caſes only) we thought we had a right to quote it.

To prop. 12. and its corollaries, their converſes are added; which, though indiſpenſable for ſolutions themſelves, are not commonly to be met with.

Prop. 13. part 2. has a new demonſtration, if it ever had any; and in 2 & 3 cor. angles of the ſpherical triangles are limited, as much as they admit of it; which does not ſeem to have been attempted by any body before

To prop. 14. & 15. their remaining parts, as well as their converſes, are added; which is neceſſary not only for the completeneſs of the ſubject, but alſo for the ſolutions themſelves.

In regard to prop. 17. there is no doubt, but many readers of ſpherics were often led by their curioſity towards it, and, as it were, miſſed it; and yet it is to be ſeen no where. Beſides other purpoſes, it is extremely uſeful in ſolutions.

Prop. 18 & 19. though they are the very foundation of the whole Spherical Trigonometry, it is very ſurpriſing, have not been demonſtrated by any body, except in one ſingle caſe: indeed it does not ſeem to be long ago, that they began to appear anew in their preſent form. V. *Keil.*

Prop. 20—23. are made to extend to all ſorts of ſpherical rightangled triangles, of which (even excluding ſuch as have more than one right angle) there are three ſpecies: for the other two angles, beſides the right one, may be either both acute, or both obtuſe, or one of them acute, and the other obtuſe: and this diverſity of theirs influences, in ſome material reſpects, the demonſtration of thoſe prop. as the attentive reader will perceive eaſily. Such weak foundations Spherical Trigonometry has had to this day! To prop. 18—23, as alſo to 14 & 15, corollaries are added concerning quadrantal triangles; a ſubject, as if it had been foreign to ſpherics, in general wholly neglected.

Of prop. 29. we have ſpoken already.

Of prop. 30. a new demonſtration was to be made, and, the ſame corrections to be applied, as in prop. 7.

In theor. 1. a new and eaſier enunciation is given, comprehending in one, what was formerly divided into two.

Theor. 2. we flatter ourſelves, may be looked upon as a conſiderable improvement towards ſolutions of ſpherical triangles; a ſmall hint of it only, in *Emerſon*, led the authors to its diſcovery. The great invention of circular elements by Lord *Neper*, is itſelf but half uſeful, unleſs accompanied with this theorem; it being generally conceived, that ſolution of obliqueangled ſpherical triangles is much more difficult than that of rightangled ones; to which, it muſt be confeſſed, bad diſpoſition of caſes, and want of their explications, and of ſufficient principles, have in a great meaſure, contributed.

III. What ſeemed particularly to deſerve explaining, and even enlarging upon, is the conſtruction of the canon, and the tables of ſolutions.

Concerning the conſtruction of the canon, it ſeemed neceſſary to exhibit it in as few propoſitions, and in a way as eaſy and plain as poſſible: and it is hoped, it is done to the ſatisfaction of learners at leaſt. The particulars of the method are chiefly from Mr. *Sharp*, in *Sherwin*'s Tables; and if ſines were actually to be calculated, and at the ſtages, propoſed in the book, to be corrected, the error from ſtage to ſtage could not be more than in 224 units of the loweſt place; and thus 9 places at leaſt out of the 12 would always remain accurate. For the contracted multiplication, we preſume, may be managed ſo judiciouſly, that in each intermediate product, and therefore in each intermediate ſine, the error ſhall not be greater than a 1 of the loweſt place; and perhaps even this is allowing too much. If a farther accuracy were wanted, the ſide of a quindecagon ought to have been aſſumed, as in prop. 7. is at preſent aſſumed the ſide of a hexagon: from that ſide would be derived the ſine of 45′, and in the ſame manner, the ſtages, where errors might be corrected, would differ by 45′; and the greateſt error from ſtage to ſtage could be at moſt but of 44 units in the loweſt place. But ſince ſome tolerable notion of the conſtruction of the canon is the only thing that is generally wanted now a days, let it ſuffice thus much to have ſaid of it. We may mention only, that the calculations in prop. 7 & 9. may be depended upon; the roots in prop. 7. having been carried to above 40 places, and compared with thoſe of Dr. *Robertſon*; and the ſines in prop. 9. having been found, beſides the way in the book, by *Newton*'s ſerieſes, to above 20 figures. Another thing perhaps worth obſerving is, that in the actual computations of triangles, in caſe arches happen to be ſmall, inſtead of the uſual way of finding any requiſite intermediate ſine by the common proportion, it would be more accurate to find it either by Mr. *Gardiner*'s method (ſee *Gardiner*'s Tables), or by prop. 5; and v. v. to find an intermediate arch from a given ſine. The ſame method had been beſt to be uſed on tangents of ſmall arches alſo; for which purpoſe a corollary is ſubjoined to that

5th. prop. Coſines and cotangents of very great arches may be found as ſines and tangents of their complements.

In regard to the Tables of Solutions, we have obſerved, that a great deal of explanation was wanting; and thought it incumbent upon us to be as explicit and ſatisfactory as poſſible. For it is to that want only, that the difficulty of ſolutions, commonly apprehended, eſpecially in ſpherics, may be aſcribed; in conſequence of which the knowledge of the higher parts of mathematics, dependent on theſe, may come to be dreaded and ſhunned. The fact is, triangles in ſeveral caſes need no ſolutions; and ſuch caſes, one ſhould think, ought to have been mentioned expreſsly, that people ſhould not be puzzled about them. In other caſes, triangles have much eaſier ſolutions than the general ones; though ſuch ſolutions may not perhaps occur to every body at every time, and therefore ought to be particularly ſet down alſo, at leaſt taken notice of. In all caſes in general, it ought to be explained, and even demonſtrated, why ſome of them are ambiguous, and other determinate. But for this it is in vain to look in authors, unleſs in part in *Segnerus*. All this therefore is performed here; and beſides, the caſes themſelves are diminiſhed in number, and properly arranged. In determinate caſes it was the care, that the ſolution ſhould be as plain as might be; and ſometimes more than one were given, when it was thought they would conduce to an eaſier and more direct diſcovery of the thing ſought. In ambiguous caſes, i. e. in caſes, which have a double ſolution, (for it may not perhaps be improper to obſerve, that the word *ambiguous* means here nothing more), it is always ſhewn, whence, and in what ſimpleſt manner the ſolution is limited; and if it is not limited, both ſolutions are explained, and the parts of each properly conformed together. It may be worth while to obſerve here to a learner, that to the end that he may become a compleat maſter of ſolving ſuch caſes, he is to remember carefully, which they are (and they are but few), and always to expect them to have two ſolutions; if, after all, they happen to have but one, as it is ſometimes ſo, to think himſelf then a gainer. In theſe caſes it has been propoſed by us, when the third ſides, or the third angles, are found, to try to limit them from prop. 8 & 9. or from prop. 13. cor. 2 & 3. provided the limitations from all other principles fail: but perhaps it may be queſtioned, whether it is poſſible to limit by the firſt, when the laſt fail: and we are obliged to leave this point in ſuſpenſe, and recommend it to be ſettled by the ſtudious and ingenious, having ourſelves never examined it ſufficiently. Certain it is, at leaſt, that conſtruction of triangles, that is, ſtereographic projection of them, can never limit indeterminate ſolutions, when theory cannot; notwithſtanding this is the opinion of Mr. *Ham* in his Appendix to *Cunn*'s *Euclid*, as well as of many others. For thus plotting ſcales would be able to ſettle relations of figures, when Geometry cannot; and plain ſcale and chord-line would ſolve plane triangles more accurately than Trigonometry itſelf; and to ſhorten the whole buſineſs of Mathematics, we need only furniſh our pockets with caſes of inſtruments, and aſſume *Philomaths* for our leaders and inſtructors, with *their pocket-companions, their guides, their keys, their compends of Mathematics*, &c. letting alone the accurate theorems and demonſtrations of *Euclid*; although it is to the early inſtruction in *them* chiefly (as M. *Montucla* juſtly obſerves in his Hiſtoire des Mathématiques) that the ſuperior and extraordinary knowledge of the *Britiſh* nation in Mathematics ought to be aſcribed. But ſurely no mechanical principles can claim preference to ſcientifical ones. *The knowledge of ſpherics is* not *mean without the knowledge of projection*; but rather it may be mean without the knowledge of preciſe ſolutions and determinations of triangles. Projection has certainly great uſes; but perhaps its application to conſtructing of triangles may be reckoned amongſt the leaſt conſiderable of them. It ſeems to be like nothing more than a picture: it is in vain to look in a picture, for what is not in the original; and the conſideration of the latter is undoubtedly always to be preferred to that of the former. Farther, as no picture can be perfect, no more can the projection of triangles; for not to mention, that it does not ſtand as yet upon ſolid foundations, and that its conſtructions, like the inſtruments and ſenſes, without which it is not able to work, cannot be very accurate, in all caſes, where the two different values of the quæſita approach one another, and perhaps in others, theory can always determine them when projection cannot.

IV. It ſeems further to be our buſineſs to ſhew, why, contrary to other authors, ſome things have been diſregarded by us and paſſed by, and other newly introduced.

Of the many relations of trigonometrical lines, to be ſeen in *Emerſon*, *Jones*'s Synopſis, *Robertſon*, &c. ſuch only have been introduced here, as are immediately neceſſary for ſolutions: the reſt, together with ſome other articles relating to Trigonometry, as inſcription of polygons in *Emerſon*, and lines of ſines, of tangents, &c. in *Maſeres*, may be much more properly collected in a ſeparate book, as being out of the province of elementary compoſitions.

Alſo, analogies, wherein addition and ſubſtraction of natural ſines, coſines &c. are uſed, though much recommended by ſome, eſpecially by the *French*, in general had not an admittance here; and the reaſon is, that their application in practice is unmanageable, and beſides not accurate enough.

In ſpherics, prop. 12. exterior angles, ſpoken of by every one, have been let alone on account of their uſeleſſneſs: they could at moſt have been ſet down in a cor. by themſelves.

Propoſitions, limiting the ſides and angles of ſome ſpherical triangles, to be found in *Emerſon*, Sect. 3. though with ſcarce a demonſtration, are alſo let alone; becauſe the caſes, wherein ſuch limitations may be applied, are determinate, and may be more generally limited otherwiſe.

There are other propoſitions, viz. of equality of ſpherical triangles, of perpendiculars drawn in them, of arches biſecting their ſides and angles; which alſo have not been admitted here. They are indeed beautiful in themſelves, although not demonſtrated accurately in general, except in *Menelaus*; but they are hardly of any uſe towards Spherical Trigonometry; and if ſometimes wanted, may be deduced from the analogies in it, as is done in the Schol. in Book II. They, with a greater propriety, belong to another part of ſpherics, little cultivated in general, which might with juſtice be called *Spherical Geometry*. Such a Spherical Geometry is not to be confounded with the drawing of triangles, commonly and undeſervedly called by that name; it being vaſtly ſuperior to it, as a ſcience. A ſmall part of it only, viz. as much as is neceſſary towards ſolution of triangles, is to be found in the *Section of Spherical Angles and Triangles*. But the whole of it, if properly arranged and completed, would conſtitute an important ſubject in Mathematics; and would be of as great uſe in Spherical Trigonometry, as the firſt ſix books of *Euclid*'s Elements are to Plane Trigonometry.

In prop. 20—23. the conſtruction is taken from *Simon Stevin* and Dr. *Simpſon*: it is not only ſimpler in itſelf, but alſo it makes the demonſtrations ſhorter; and probably the analogies will be the more eaſily remembered. In the common conſtructions, arches are produced farther; and in Dr. *Keil's*, even more ſo than is neceſſary. And indeed, if from the poles B and C (See Pl. 2. Fig. 15.) arches of great circles be deſcribed, and produced, that they may meet the hypotenuſe and the ſide on which the pole is, each in two points, and the ſide oppoſite to the pole in one point, there would be formed three other ſuch triangles, as CEF; and from each of them four analogies may be deduced, as alſo from the triangle ABC: but all theſe analogies are in reality no other than the common ſix on rightangled triangles. We know, it is recommended by ſome, inſtead of remembering the analogies, or looking for them in the triangle CEF only, to produce the ſides conſtantly, and thus by marking the data and quæſita on triangles, to hunt the analogies out: in ſuch a caſe, a field opens itſelf large enough in conſcience to loſe oneſelf in; eſpecially if the arches are not drawn accurately enough to come again and ſtop at the triangle ABC.—In obliqueangled triangles alſo, other ways of conſtruction and ſolution had been uſed, as producing ſome of the arches to quadrants, and finding the value of the perpendicular conſtantly: but both theſe are not conducive to any eaſineſs either in calculations or limitations. Theſe, and many other ſhifts of ſtill lower a nature (as uſing the ſame letters, marking the right angle by the letter A conſtantly, making particular ſides to be baſes and perpendiculars, endeavouring to eſcape cotangents, &c.) had been invented, that learners might be humoured, and grow as inattentive and regardleſs of the true buſineſs of ſpherics as poſſible. But for a rational and actual eaſe to memory, there can be no better helps, than theor. 1 & 2.

As ſome things have been rejected here, ſo other have been added, whoſe uſe and application may not ſeem immediate. Such are in Book I. def. 7. def. 8. cor. 2. prop. 10. the laſt part. Theſe will be found to be neceſſary for the doctrine of projections. In Book II. def. 4. cor. 2. prop. 4. and its corr. prop. 5. cor. have been introduced, that they may ſerve for the ſame projections, as well as for the purpoſes of Aſtronony and Navigation. For ſuch has been the ſtate of ſpherics, that three ſciences, *of Circles of the Sphere*, *of Spherical Geometry* and *Spherical Trigonometry*, all different ones, one ſhould think, have been conſtantly blended, and looked upon as one ſcience; and even that one has been ſo very incomplete,

plete, that in other higher ſciences, things belonging to it are always conſidered as known previouſly (though no one can tell whence) or muſt be introduced there as ſtrange lemmas. Some other things are added, that the ſubject may become complete, e. g. in Book II. prop. 8. cor.; prop. 25. cor.; theor. 2. cor. And in regard to the Appendix, it has been ſubjoined, that, among other purpoſes, it may keep up that beautiful harmony, in which the two Trigonometries ſo evidently agree.

V. To ſum up the whole of what was advanced thus far, we fancy we may ſet down here in brief, what has been improved, and what has been invented by us: and we hope, that herein the candid and liberal will not look upon us as vain and preſumptuous, but rather as juſt to others without being unjuſt to ourſelves.

Among what has been improved, may be reckoned in Book I. def. 2; def. 3.; 4. def. 1. cor.; 8. def. 2. cor.; 9. def. 1, 2, 5, 6. cor.; prop. 1. cor.; prop. 2.; prop. 4. & ſchol.; prop. 5.; prop. 6 & corr. 2 & 4.; prop. 7; prop. 9.; lem.; prop. 11.; prop. 12 & cor.; prop. 13 & 2 cor.; prop. 14. & 3 cor.; prop. 15.; ranging of the caſes; in oblique-angled triangles caſe 1.; caſe 3.; caſe. 5. In Book II. def. 2.; def. 3.; corr. 1, 2, 3, 5.; def. 8.; prop. 1 & corr.; prop. 2 & corr.; prop. 3.; prop. 4. cor. 3.; prop. 5 & cor.; prop. 7.; prop. 8.; prop. 9.; prop. 11.; prop. 13. part. 2d and its 2 & 3. cor. 3d. parts; prop. 14.; prop. 18, 19, 20, 21, 22, 23, 24, 25, 26; theor. 1.; Tables of Solutions throughout; theor. 3. after *Baker*; theor. 5. cor. 3. after *Emerſon*; theor. 6. & cor. after *Emerſon*.

Among what is our own, we think, may be placed in Book I. def. 1.; def. 3. cor.; def. 5. cor.; def. 9. cor. 4.; prop. 2. cor.; prop. 4. cor.; prop. 5. cor.; prop. 9. ſchol.; prop. 11. cor.; prop. 14. cor. 4, & cor. 5. demonſtr.; ſchol. to ſolut. In book II. def. 1.; prop. 2. corr. 3, 4, 6.; prop. 6. demonſtr. and corr. 2, 3. after *Segnerus*; prop. 7, enunciation & cor.; prop. 8. cor.; prop. 10. part 1. demonſtr.; prop. 12 & corr. converſes; prop. 14. cor.; prop. 15. caſe 3 & cor. 4.; prop. 17.; prop. 18—23. corr.; prop. 29. demonſtr. & cor. 4, 5.; prop. 30. demonſtr. & cor. 4, 5.; theor. 2 & cor.; ſchol. to ſolutions, but towards the end from *Segnerus*; lem. 1. part. 2. demonſtr.; lem. 2. demonſtr.; lem. 3. part 2, & demonſtr.; theor. 4. cor. 6.; theor. 5. cor. 2.

VI. As we have applied ourſelves to the ſtudy of this part of Mathematics with much attention, we hope to be pardoned, if we ſuggeſt here ſomething towards its improvement. We muſt not indeed flatter ourſelves, that we have already arrived at the ſummit of perfection in Mathematics: much there is remaining to invent and to improve, and that not only for theoretical purpoſes, but even for the practical. Inſtances of this might be eaſily given from every ſingle part of thoſe ſciences. And this conſideration alone ſhould excite us to exert our talents in theſe rational and noble ſciences: and we may very well have before our eyes that period, glorious for the *Britiſh* nation, we mean, the end of the laſt century, and the beginning of the preſent, wherein ſo many geniuſes contended in emulation, and made in the algebraical part ſuch conſiderable improvements.

In the parts before our conſideration, it would be perhaps an improvement, inſtead of prop. 25—28. of Book II. to introduce theorems 3 & 4, inventions of Lord *Neper*, by way of propoſitions. The firſt of them ſhould be demonſtrated by means of planes, as is done in prop. 18, 19, & 29. As yet no ſuch demonſtration is to be found any where, that in *Caſwell* uſing projection; though it may be perhaps of great aſſiſtance towards it. The other theorem would be deduced from this, by means of ſupplemental triangle. Solutions derived from them would be applied, as is explained in theor. 4. cor. 1—4; and Spherical Trigonometry would be in all reſpects like the Plane.

Another improvement would be, if we could find all the quæſita by one ratio, or though by ſeveral, yet ſuch only that their terms could be derived from one another. Inſtances of the former are in plane triangles, ſecond caſes of both rightangled and obliqueangled; and inſtances of the latter are caſes 1ſt. and 4th. of the rightangled, and caſe 5th. and perhaps caſe 4th of the obliqueangled. But in ſpherics no ſuch thing at all appears. We have endeavoured as much as was in our power to compaſs ſuch ſolutions, and with that aim diſcovered and introduced the cor. prop. 11. but to find more of them may require more, and perhaps new principles. To this we may add the finding of any one quæſitum immediately from the data, which is not ſo at preſent in caſe 3. of plane rightangled triangles, in caſe 1ſt. of the obliqueangled, and in moſt caſes of obliqueangled ſpherical triangles, except by two analogies. It was with this view that we have added the laſt part of caſe 3. of plane obliqueangled triangles.

Another thing ſerving towards perfection of Trigonometry, would be the examination of the poſſibility and impoſſibility of the data of triangles, which are ſometimes not eaſily diſcoverable, and particularly in ſpherics. This would beſides ſerve towards accuracy and ſurety of ſome practical operations, more than one may at firſt ſight imagine. But this perhaps may be deduced from Mr. *Cotes*'s Æſtimatio Errorum in mixta Matheſi.

It might alſo greatly conduce to promoting of this part of Mathematics, if from any logarithmic ſine of an arch being given, we could immediately deduce its logarithmic coſine or tangent, and from coſine of an arch its ſine or tangent, and laſtly from tangent of an arch its ſine or coſine. But this is the caſe only on right and verſed ſines, and on tangents and cotangents at preſent. It would ſerve to prevent multiplication of inaccuracies in ſolutions; though perhaps may not be ſo eaſily compaſſed about as one would deſire.

We may here add a few words concerning the extent of Trigonometry. In every plane triangle there being three ſides and three angles; three perpendiculars drawn upon the ſides, and the ſegments of thoſe ſides, as alſo the verticle angles made by the perpendiculars; three ſtrait lines biſecting the angles, or in general dividing them in a given ratio of angles, ſines of angles, &c. and the ſegments of the oppoſite ſides, as alſo the angles at them, made by the biſecting or dividing lines; three ſtrait lines biſecting the ſides, or in general cutting them in any ratio, or drawn to them in any given angle, and the ſegments of thoſe ſides, as alſo the vertical angles made by them; the perimeter of the triangle, and its area; and the ſum, the difference, product, quotient or ratio, power, root, of one, or two, or or more of theſe: And the ſame being in a ſpherical triangle, *mutatis mutandis*;—it is the buſineſs of Trigonometry from any three of theſe to find the remaining, when poſſible. Some few of theſe may be found by propoſitions in *Emerſon*'s and *Thoſ. Simpſon*'s Trigonometries; and a great many are to be ſeen in the laſt author's Appendix to Algebra, as indeed in moſt Algebras, and in the late extracts from *Ladies Diary*.—Trigonometrical ſolutions may alſo be extended to quadrilateral and other multilateral figures, plane as well as ſpherical, as may be ſeen in *Simon Stevin*. Several of theſe ſolutions both on triangles and other figures have not an inconſiderable uſe even in practice; and it were to be wiſhed, that for purpoſes of ſcience in general, they were collected by ſome ingenious hand, and exhibited geometrically, trigonometrically, and algebraically. It is pity, that the author of *Synopſis of the data of a triangle* did not think of the matter in this light; and his referring us to ſo many books and ſmall pieces, which are hard to be got, can afford but little aſſiſtance.

VII. It remains only to be added, that the reader, for his eaſier and completer comprehenſion, is deſired ſometimes to draw figures for himſelf. This need be done chiefly in ſpherics, and particularly in def. 3. corr.; def. 4. cor.; prop. 1. corr.; prop. 2. & corr.; prop. 20—23 (of which we have mentioned before), and perhaps in ſome few others. It is hoped alſo, it will not be taken amiſs, that definitions of the firſt and eleventh book of *Euclid*, and alſo the axioms, are ſometimes quoted by us. The reaſon is, that theſe things are apt to be forgotten; and that therefore demonſtrations, dependent on them, though ever ſo accurate, are in that caſe worth nothing. It appears from the books of the ancients, that they very ſeldom, if ever, quoted any previous principles in their mathematical treatiſes: but it may be proved by the experience of ſeveral modern ages, that the ſimpleſt things are not the moſt familiar things to us: and this alone has been a ſource of a great many errors, as well as of perplexities and falſe improvements in Mathematics. It is to be remembered alſo, that the Elements of *Euclid*, quoted here, are thoſe of Dr. *Simpſon*'s edition only.

If it may ſerve as a recommendation to this book, it may be mentioned farther, that the authors have at ſeveral different times ſpent more than ſeven years about it; that they had taken ſeveral opportunities of reviſing and correcting it, it having been written over ſeven times in that period by the hand of one of the authors, and read at different times to above 300 learners, who were afterwards examined in it ſtrictly, this may perhaps be conſidered as ſome editions of the book.

And now the authors have only earneſtly to beg of the good *Britiſh* nation its wonted candour and indulgence to this work. They are ſenſible of their many inaccuracies in the ſtile and language, for which they intreat pardon. If their pains be kindly regarded, it will be a great encouragement to them to offer to the public ſomething farther. They mean in particular an edition of *Euclid*'s Elements, viz. of books 1—6, & 11th & 12th, and an additional book concerning regular figures both plane and ſolid, extracted from the remaining books of the elements and from other authors.

They will adjoin there notes critical and explanatory, both from Dr. *Simpſon* and Mr. *Williamſon*, as well as of their own; and will ſtudy to obſerve the ſame generality, completeneſs, accuracy, and preciſion, that they have endeavoured to do in theſe two Trigonometries. After this they intend to attempt an edition of *Archimedes*'s two books *De Sphaera et Cylindro*, and then to go on to other parts of Mathematics, it being their wiſh to complete one whole ſyſtem, which at preſent ſeems to them to be wanting.

Cronſtadt,
Oct. 1, 1786.

ELEMENTS
OF
TRIGONOMETRY,
BOTH
PLANE AND SPHERICAL.

BOOK I.

Plane Trigonometry.

DEFINITIONS.

Book I.

1. IT appears from the Elements of Euclid, that angles at the center of a circle have the ſame ratio to four right angles, as the arches, on which they ſtand, to the whole circumference: for this reaſon, both the circumference of a circle and four right angles are wont to be divided into the ſame number of equal parts, of which as many are reckoned in any angle, as in the arch on which it ſtands at the center of the circle. And becauſe arches may be known from certain ſtrait lines given in poſition; it is generally underſtood, that thoſe ſtrait lines belong as well to the arches, as to the angles at the center, which ſtand on them.

2. The circumference of every circle is ſuppoſed to be divided into 360 equal parts, which are called degrees (μοιραὶ, gradus); and every degree into 60 equal parts, which are called firſt minutes, or minutes (λεπτὰ,

minuta, and minuta prima); and each minute into 60 equal parts, called ſecond minutes, or ſeconds (minuta ſecunda); and thus farther on the ſexageſimal diviſion is ſuppoſed to continue. Hence an arch, as well as the angle at the center of a circle, ſtanding upon it, are ſaid to be, e. g. of 30 degrees, 15 minutes, 45 ſeconds, &c. and this is commonly written thus: 30° 15′ 45″, &c.

Corollary. Whatever be the radius of a circle, on ſome arch of which an angle ſtands at the center of it; that arch will be always of the ſame number of degrees, minutes, ſeconds, &c. as appears from def. 1 & 2.

3. The difference between half the circumference of a circle, and any of its arches, as alſo the difference between two right angles, and any angle or angles, are called the ſupplements of that arch, or of that angle. But the difference between a quarter of the circumference, or a quadrant, and an arch which is leſs than half the circumference; or the difference between three quadrants and an arch which is between half the circumference and the whole of it, is called the complement of either of thoſe arches. Thus alſo are called the differences between one or three right angles, and any angle or angles.

Cor. Any arch or angle, and their ſupplements, have the ſame complement.

4. The right ſine, or, more commonly, the ſine of an arch, or of an angle, is called the perpendicular ſtrait line drawn from one end of the arch upon the diameter that paſſes through the other end of that arch.

Cor. 1. Half the chord of any arch is equal to the ſine of half that arch.

Plate I.
Figure I.

Let A B C be a circle, and D its center; and let A B be the chord of the arch A B, and B E half the chord A B. Biſect the arch A B in F, and join F E. It is manifeſt from El. 8. 1. that the angles B E F, F E A are equal and right: therefore the center D is in the ſtrait line F E [a], produced if neceſſary; and ſo B E is the ſine of the arch B F.

Cor. 2. The ſine of a quadrant, or of a right angle, is equal to the radius [b], which is called alſo the total ſine; and is the greateſt of all ſines [c].

[a] El. cor. 1. 3.

[b] 1 cor.

[c] Ib. 15. 3.

5. The verſed ſine of an arch, or of an angle, is called that ſegment of the diameter paſſing through one end of the arch, which lies between this end and the ſine drawn from the other end of the arch.

COR. The verſed ſine of a quadrant, or of a right angle, is equal to the radius; as appears from 2 cor. 4 def.

6. The tangent of an arch, or of an angle, is called the ſtrait line, which touches the circle at one of the ends of the arch, but is continued no farther than till it meet the diameter produced that paſſes through the other end of the arch.

7. Theſe mitangent, or half tangent, of an arch, or of an angle, is called the tangent of half that arch, or of half that angle.

8. The ſecant of an arch, or of an angle, is called that ſegment of the diameter paſſing through one end of the arch and produced, which lies between the center and the ſtrait line touching the circle at the other end of the arch.

COR. 1. Hence it appears, that the ſecant of every arch, or angle, is greater than the radius; and that the tangent, as well as the ſecant, of a quadrant, or of a right angle, are infinite.

Figure I.

COR. 2. The ſemitangent AG of an arch AB is equal to the ſtrait line DH, which is drawn from the center at right angles to the diameter paſſing through one of the ends, A, of that arch, and is produced till it meet the chord of the ſupplement, CB, that paſſes either through the ſame, or through the other end, B, of the arch AB.

For the angles DAG[a], CDH are right ones, and the angle ADG is equal to the angle DCH[b], and DA is equal to DC; therefore AG alſo is equal to DH[c].

[a] El. 18. 3.

[b] 7. def. & El. 20. 3.

[c] El. 26. 1.

9. The ſine, the verſed ſine, the tangent, and the ſecant of the complement of any arch or angle, are called the coſine, the coverſed ſine, the cotangent, and the coſecant of that arch itſelf, or of that angle itſelf.

COR. 1. If a diameter be drawn through one of the ends of an arch, its ſegment which lies between the center and the ſine drawn from the other end of the arch, is equal to the coſine of that arch; as is manifeſt from El. 34. 1.

Cor. 2. Half the chord of the ſupplement of any arch, is equal to the coſine of half that arch.

Figure I.

Let the arch B C be the ſupplement of the arch A B; and let their chords be B C, A B. Biſect the arch A B and the chord B C in F, K; and join D F, D K. And becauſe the angle C B A is a right one[a], and ſo is the angle D K B[b]; D K is parallel to A B[a]. Alſo, becauſe the angle C B A is a right one[a], and ſo is the angle B E D[d]; B C is parallel to D E[c]: wherefore E K is a parallelogram, and B K is equal to D E[e]. But becauſe B E D is a right angle, B E is the ſine of the arch B F[f], and D E its coſine[g]: therefore B K is equal to the coſine of the arch B F.

[a] El. 31. 3.
[b] El. 3. 3.
[c] El. 28. 1.
[d] El. 4. 1.
[e] El. 34. 1.
[f] 4. def.
[g] 1. cor.

Cor. 3. The verſed ſine of an arch, or angle, which are leſs than 90°, is equal to the difference between the radius and the coſine[g] of that arch, or angle: but the verſed ſine of an arch, or of an angle, which are between 90° and 180°, is equal to the ſum of the radius and the coſine[g] of that arch or angle.

Cor. 4. If in the ſame, or equal, circles, arches, or angles at the centers, be equal; or if an arch be to the circumference, as an angle at the center ſtanding on ſome other arch, is to four right angles; the ſines, the verſed ſines, the tangents, the ſecants, &c. of thoſe arches, or angles, will be equal; as is manifeſt from the elements.

Cor. 5. If equal angles ſtand at the centers of unequal circles; their ſines, verſed ſines, tangents, ſecants, &c. will be to one another, as the radii of thoſe circles.

Figure II.

Let the angles A C B, D C E be equal; and let the angle A C B, which is in the circle A B, have the ſine B F, the verſed ſine F A, the tangent A G, the ſecant G C; and the angle D C E, which is in the circle D E, have the ſine E H, the verſed ſine H D, the tangent D K, the ſecant K C. Becauſe the angle B F C is equal to the angle E H C[f], and the angle C A G to the angle C D K[h]; and the angles A C B, D C E are equal alſo; the triangles B C F, E C H, as alſo A C G, C D K, are equiangular: therefore the ſine B F is to the ſine E H, as the radius B C to the radius C E[k]; and the coſine C F[g] to the coſine C H[g], as the radius B C, or C A, to the radius C E, or C D[k]: wherefore alſo the verſed ſine F A[l] is to the verſed ſine H D[l], as the radius C A to the radius C D[m]. Alſo,

[h] 6. def. & El. 18. 3.
[k] El. 4. 6.
[l] 3. cor.
[m] El. 19 or 12. 5.

the tangent A G is to the tangent D K, and the ſecant G C to the ſecant C K, as the radius A C to the radius C D[k]. And the ſame is manifeſt on the coverſed ſines, cotangents, and coſecants; ſince the complements of equal angles are equal.

Cor. 6. Any arch A B, or an angle A D B, and their ſupplements, the arch B C, and the angle B D C, have the ſine B L, the tangent A M, the ſecant M D, the coſine B N or D L[g], the coverſed ſine N O, the cotangent O P, and the coſecant P D, all the ſame: but their verſed ſines A L, C L are different, (except the arch, or the angle, be of 90° or 270°); and either of theſe verſed ſines is the difference between the other and the whole diameter.

Figure I.

This is manifeſt both on the right and the verſed ſines. Produce now B D to R; and A M will be the tangent[n], and M D the ſecant[o], of the arch A R, or of the angle A D R: wherefore alſo, they will be the tangent and the ſecant of the arch B C, or of the angle B D C[p]. And on the coſines, &c. it appears from cor. 3 def.

10. The trigonometrical canon is a table of numbers, wherein the radius of a circle is commonly ſuppoſed to be divided into 10,000,000,000 equal parts; and it is ſhewn what number of thoſe parts is contained in the ſine, the tangent, the ſecant, the verſed ſine, &c. of every arch, or of every angle, of that circle. By means of this canon, an arch, or an angle, being given, their ſines, tangents, &c. are found; and converſely, a ſine, or a tangent, &c. being given, the arch and the angle, to which they belong, are found.

[k] El. 4. 6.

[g] 1. cor.

[n] 6. def.

[o] 8 def.

[p] 4. cor.

Of Trigonometrical Lines in general.

Theorem.

PROPOSITION I.

THE coſine of any arch, or angle, is to their ſine, as the radius to the tangent, and as the cotangent to the radius: and the coſine is to the radius, as the radius to the ſecant, and as the cotangent to the coſecant: and the ſign is to the radius, as the tangent to the ſecant, and as the radius to the coſecant.

This appears from the equiangular triangles D B L or D B N, D M A, and D O P, in fig. 1.

Cor. The radius is a mean proportional between the tangent and the cotangent of any arch, or angle, as alſo between the coſine and the ſecant, and between the ſine and the coſecant. Wherefore the tangent and the cotangent of an arch, or angle, are reciprocally proportional to the tangent and the cotangent of any other arch, or angle, in the ſame circle; and the coſine and ſecant of that to the coſine and ſecant of this, as alſo the ſine and coſecant of the firſt to the ſine and coſecant of the other[a].

[a] El. 23. 5.

Theor.

PROPOSITION II.

The ſine of half an arch is a mean proportional between half the radius and the verſed ſine of the whole arch, as alſo between the tangent of half the arch and half the ſine of the whole arch: and the coſine of half an arch is a mean proportional between half the radius and the verſed ſine of the ſupplement of the whole arch, as alſo between the cotangent of half the arch and half the ſine of the whole arch.

Figure I.

Let the arch A F be the half of the arch A B, and B L the ſine of the arch A B: wherefore A L, L C are the verſed ſines of the arch A B and of its ſupplement B C[a]. Draw A B, B C, and D F meeting A B in E; draw alſo D O at right angles to A C; and let A G be the tangent of the arch A F, and O S its cotangent.

[a] 5. def.

Therefore A E is the half of A B[b], and is the ſine of the arch A F[c]; and D E is its coſine[d], and is equal to half B C[e]. And becauſe the triangles A D E, A B L, A E G are equiangular, it will be as half D A to A E, ſo half A B, i. e. the ſame A E, to A L[f]; and as A G to A E, ſo half A B, i. e. the ſame A E, to half B L[f].

[b] El. 4. 1.
[c] 1. cor. 4. def.
[d] 1. cor. 9. def.
[e] 2. cor. 9. def.
[f] El. 4. 6. & 15. 5.

Alſo, the triangles A D E, B C L, D O S are equiangular: and ſo half D A is to D E, as half B C, i. e. the ſame D E, to C L[f]; and O S is to half D O, or half D A, as C L to half B L[f]: and therefore, ex aequali perturbatè, O S is to D E, as D E to half B L[g]. Q. E. D.

[g] El. 23. 5.

Cor. The verſed ſines of an arch and of its ſupplement are to one another, as the tangent and the cotangent of half that arch.

Becauſe $AE^2 = \frac{1}{2} DA \times AL$[h] $= AG \times \frac{1}{2} BL$[h], and $DE^2 = \frac{1}{2} DA \times CL$[h] $= OS \times \frac{1}{2} BL$[h]; it will be $\frac{1}{2} DA \times AL : \frac{1}{2} DA \times CL :: AG \times \frac{1}{2} BL : OS \times \frac{1}{2} BL$; therefore $AL : CL :: AG : OS$[k].

[h] prop. & El. 17. 6.
[k] El. 1. 6.

PROPOSITION III. Theor.

The ſine of the arch of 30° is equal to the half of the radius.

For the ſide of an equilateral and equiangular hexagon inſcribed into a circle, i. e. the chord of the arch of 60°[a], is equal to the radius of that circle[b]: and half the chord of the arch of 60° is equal to the ſine of the arch of 30°[c]: therefore the ſine of the arch of 30° is equal to half the radius.

[a] 2 def.
[b] El. cor. 15. 4.
[c] 1. cor. 4. def.

PROPOSITION IV. Theor.

The ſine of the arch of 45° is equal to its coſine, and its tangent is equal to the radius.

For the complement of the arch of 45°, is an arch of 45° alſo: wherefore the ſine of the complement, i. e. the coſine, of the arch of 45° is equal to the ſine of it[a]. And becauſe the ſine of an arch is to its coſine, as the tangent to the radius[b]; and the ſine of the arch of 45° is equal to its coſine; therefore alſo its tangent is equal to the radius[c].

[a] 4. cor. 9. def.

[b] 1. 1.

[c] El. A. 5.

Cor. And from El. 19. 1. it appears, that the tangent of an arch leſſer than 45°, is leſſer than the radius; and the tangent of an arch greater than 45°, but leſſer than 90°, is greater than the radius. Wherefore, converſely, in arches which are leſſer than quadrants, the tangent which is leſſer than the radius, is the tangent of an arch leſſer than 45°; and that which is greater than the radius, is the tangent of an arch greater than 45°.

SCHOLIUM.

Sines of arches, or angles, of the ſmalleſt length in the beginning, increaſe orderly to the length of the radius, to which they become equal at 90°[a]; from 90° to 180° they decreaſe, ſo that at laſt the arch of 180° has no ſine: from 180° to 270° they increaſe again, juſt as before 90°; and from 270° to 360° they decreaſe again, juſt as from 90° to 180°. Coſines, equal to the radius in the beginning, decreaſe orderly as far as 90°, ſo that at laſt the arch of 90° has no coſine; from 90° to 180° they increaſe orderly, till at 180° they become equal to the radius: from 180° to 270° they proceed in the ſame manner, as before 90°; and from 270° to 360°, as from 90° to 180°.

[a] 2. cor. 4 def.

Verſed ſines, of the ſmalleſt length in the beginning, increaſe orderly as far as 180°, the verſed ſine of 90° being equal to the radius[b], and that of 180° being equal to the diameter: from 180° to 360°, beginning from the length of the diameter, they decreaſe orderly. Coverſed ſines, of the length of the radius in the beginning, decreaſe orderly as far as 90°, ſo that the arch of 90° has no coverſed ſine; from 90° to 180° they increaſe, and the coverſed ſine of 180° is equal to the radius: from 180° to 360°, they proceed juſt as before 180°.

[b] cor. 5. def.

Tangents, of the ſmalleſt length in the beginning, increaſe orderly as far as 90°, the tangent of 45° being equal to the radius[c], and that of 90° being infinite[d]; from 90° to 180° they decreaſe, the tangent of 135° being equal to the radius[e], and the arch of 180° having no tangent: the ſame order

[c] 4. 1.

1. cor. 8. def.

[e] 4. 1. & 6. cor. 9. def.

order they obſerve from 180° to 360°. Cotangents obſerve the contrary order; they firſt decreaſe, and then increaſe.

Secants, of the length of the radius in the beginning,[d] increaſe orderly as far as 90°; ſo that the ſecant of this arch is infinite [d]; from 90 to 180° they decreaſe, the ſecant of 180° becoming equal to the radius: the ſame order they follow from 180° to 360°. Coſecants follow the contrary order; they firſt decreaſe from infinity to the length of the radius, then increaſe from the length of the radius to infinity.

[d] 1. cor. 8. def.

All this may be demonſtrated by means of El. 4. 6. and cor. 13. and prop. 10. book 5. and Trig. 6. cor. 9. def.

The Trigonometrical Canon.

PROPOSITION V.

Theor.

SINES of very ſmall arches, in the ſame circle, are to one another nearly as the arches themſelves.

Let A B, B C, in the circle A B C, be very ſmall arches, and let their ſines be AE, C F. Draw BG the tangent of the arch A B, and through A draw the ſtrait line A H parallel to the radius D B.

Figure III.

Becauſe the triangles A H G, D B G are equiangular [a]; as A H, or E B [b], to D B, ſo is H G to G B [c]. But ſince the arch A B is ſuppoſed ſmall, the end E of its ſine A E nearly coincides with the point B, ſo that E B is very ſmall in compariſon of the radius D B: wherefore alſo H G is very ſmall in compariſon of B G [d]. Again, ſince the arch A B is very ſmall, its ſine A E and the tangent B G are alſo very ſmall in compariſon of the radius D B [e]: therefore there are two ſtrait lines, B G and A E, or B H [b], which are very ſmall in compariſon of the radius D B; and they have their difference H G very ſmall even in compariſon of B G: and therefore the ſine A E and the tangent B G may be reckoned nearly

[a] El. 29. 1.
[b] El. 34. 1.
[c] El. 4. 6.
[d] El. D. 5.
[e] Sch. 4. 1.

equal in compariſon of the radius D B. But the arch A B is leſſer than its tangent B G, and greater than its ſine AE: (for if the arch B K, which is double of the arch A B, be taken; and the ſtrait line G K, and the chord B K be drawn; then the arch B K is leſſer than the two tangents B G, G K together, but greater than the chord B K. Archim. Sph. Cyl. 2 & 1. 1.): much more nearly equal may therefore the arch A B and its ſine A E be reckoned in compariſon of the radius D B. After the ſame manner, becauſe the arch B C is alſo ſuppoſed very ſmall, it may be ſhewn, that the arch B C and its ſine C F may be reckoned as equal in compariſon of the radius D B. Therefore, when the arches A B, B C are very ſmall, it is nearly as the arch A B to the arch B C, ſo the ſine A E to the ſine C F[f].

Cor. In like manner it may be ſhewn, that tangents of very ſmall arches are alſo to one another nearly as the arches themſelves.

Theor.

PROPOSITION VI.

If three arches of a circle be equidifferent, the chord of the mean is to the two chords of the extremes, as the radius to the chord of the ſupplement of the common difference: and if the greateſt of the arches be not greater than a ſemicircumference, the chord of the ſupplement of the mean alſo will be to the two chords of the ſupplements of the extremes in the ſame ratio.

Figure IV. Let the arches A B, A C, A D, in the circle A C E, have their differences B C, C D equal; and of theſe arches and their ſupplements B E, C E, D E, let the chords A B, A C, A D, and B E, C E, D E be drawn; as alſo the diameter A E; and let there be taken an arch E F, equal to either of the arches B C, C D: therefore the arch E F is the common difference of the arches A B, A C, A D; and the arch A F is the ſupplement of E F: let the chord A F of the arch A F be drawn. I ſay, that as the radius A G is to A F, ſo is A C to A B and A D together; and that if A D is not greater than a ſemicircumference, in the ſame ratio is C E alſo to B E and D E together.

Join

[f] El. 7. & 11. 5.

Join G F, B C, C D; and having produced A D, E B, draw from C to A D; C H equal to C A, and from C to E B draw C K equal to C E.

Becauſe the quadrilateral A B C D is in a circle, its oppoſite angles A D C, C B A are equal to two right angles[a]. But alſo the adjacent angles A D C, C D H are equal to two right angles[b]: therefore the angle C B A is equal to the angle C D H. And becauſe the arch B C is equal to the arch C D, the angle B A C is equal to the angle C A D[c]; but the angle C A D is equal to the angle D H C, becauſe H C is equal to C A[d]: therefore the angle B A C is equal to the angle D H C. Wherefore two angles C B A, B A C of the triangle A B C are equal to two angles C D H, D H C of the triangle C D H, each to each; and the ſide A C is equal to the ſide C H, thoſe that ſubtend equal angles: therefore A B is equal to D H[e]. Add in common A D; and A B, A D together are equal to A H. In the ſame manner it is demonſtrated, that B E, D E together are equal to E K. And becauſe the angle G A F is equal to each of the angles C A H, C E K[c]; the iſoſceles triangles A F G, A C H, E C K are therefore equiangular[d]: wherefore as G A to A F, ſo is A C to A H[f], i. e. to A B, A D together; and ſo is C E to E K[f], i. e. to B E, D E together. Q. E. D.

COR. 1. And if the arch A B be equal to the difference B C; as the radius A G to the chord A F, ſo will be the chord A B alſo to the chord A C; and ſo will be B E, the chord of the ſupplement of the arch A B, to the diameter A E together with C E, the chord of the ſupplement of the arch A C.

For if to E A produced there be drawn from B the ſtrait line B L equal to B E; A L will be equal to C E[e]; and the triangles A B C, B E L will be equiangular to the triangle A F G.

COR. 2. The ſame things being ſuppoſed, as in the prop. and cor. 1. if the greateſt of the arches be not greater than a ſemicircumference; as the radius to twice the coſine of the common difference, ſo is the ſine of the arch A B to the ſine of the arch A C, and the ſine of the arch A C to the two ſines of the arches A B, A D together: and if the greateſt of the arches be not greater than a quadrant, in the ſame ratio will the coſine alſo of the arch A B be to the radius together with the coſine of the arch A C, and alſo the coſine of the arch A C to the two coſines of the arches A B, A D together.

[a] El. 22. 3.

[b] El. 13. 1.

[c] El. 27. 3.

[d] El. 5. 1.

[e] El. 26. 1.

[f] El. 4. 6.

For if the double of each of theſe arches, as alſo of their difference, be taken; the double arches will alſo be equidifferent; and the chord of the ſupplement of the difference of the double arches will be equal to twice the coſine of the difference of the arches A B, A C, A D themſelves[g]. But the chords of the double arches and of their ſupplements have this ratio to one another, by the prop. and cor. 1. therefore alſo the ſines and the coſines of the arches A B, A C, A D themſelves, being the halves of thoſe chords[h] and of the chords of the ſupplements[g], will have the ſame ratio to one another[k].

Cor. 3. Hence it appears, that when the difference of the three arches A B, A C, A D is an arch of 60°, the ſine of the mean A C is equal to the two ſines of the extremes A B, A D together.

For the radius is then equal to twice the coſine of the difference, i. e. to twice the coſine of 60°, or twice the ſine of 30°[l].

Cor. 4. If of three equidifferent arches B M, M A, A C, the mean M A be of 60°; the ſine of the greater of the extremes, A C, is equal to the ſine of the leſſer M B, together with the ſine of the common difference B A, or M C.

Take the arch C D of 60°. And becauſe M C is equal to A B, add B M: then B C is equal to A M, and ſo it is an arch of 60°. But ſo is the arch C D alſo: therefore the remaining two A B, D E are together of 60°, and therefore are equal to the arch A M, or to the two A B, B M together, take away the common A B: and the arch D E is equal to the arch B M, and their ſines are equal[m]. But the ſine of D E is the ſame as that of its ſupplement A D[n]: therefore the ſine of A D is equal to the ſine of B M. And becauſe each of the arches B C, C D is of 60°; the three arches A B, A C, A D are equidifferent, and their common difference is an arch of 60°: therefore the ſine of A C is equal to the two ſines of A D, A B together[p], and therefore is equal to the two ſines of M B, A B together.

Problem.

PROPOSITION VII.

In a circle, whoſe radius is 1, to find the ſine of an arch of one minute.

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[g] 2. cor. 9. def.

[h] 1. cor. 4. def.

[k] El. 15. 5.

[l] 3. 1.

[m] 4. cor. 9. def.

[n] 6. cor. 9. def.

[p] 3. cor.

BOOK I.
Figure V.

Let the radius A C of the circle A D B be 1: therefore the diameter A B is 2. Take the arch B D of 60°, and join D A, D B. Biſect the arch B D in E′, and join E′A, E′B: then biſect the arch E′B in E″, and join E″A, E″B: alſo biſect the arch E″B, as alſo its half, and its quarter, and ſo on forwards; and from the points of ſections draw ſtrait lines to the ends of the diameter A B.

Becauſe D B is the chord of an arch of 60°, it is equal to the radius [a]; and ſo D B is 1: therefore $AD\ (=\sqrt{AB^2-DB^2}\,^{b}) = \sqrt{4-1} = \sqrt{3}$. And becauſe the arches A D, A E′, A B are equidifferent, and B E′ is their difference; as the radius A C to A E′, ſo is A E′ to A B, A D together [c]: and therefore $AE'\ (=\sqrt{AC\times\overline{AB+AD}}\,^{d}) = \sqrt{2+AD}$. For the ſame reaſon, becauſe the arches A E′, A E″, A B are equidifferent, and B E″ is their difference; as the radius A C to A E″, ſo is A E″ to A B, A E′ together [c]: and therefore $AE''\ (=\sqrt{AC\times\overline{AB+AE'}}\,^{d}) = \sqrt{2+AE'}$. In this manner there will be found,

$$\begin{array}{lllll}
 & AD & =\sqrt{3}\ \ldots & =1.732050807569, & DB \\
 & AE^{i} & =\sqrt{2+AD} & =1.931851652578, & E^{i}B \\
 & AE^{ii} & =\sqrt{2+AE^{i}} & =1.982889722748, & E^{ii}B \\
 & AE^{iii} & =\sqrt{2+AE^{ii}} & =1.995717846477, & E^{iii}B \\
 & AE^{iv} & =\sqrt{2+AE^{iii}} & =1.998929174953, & E^{iv}B \\
 & AE^{v} & =\sqrt{2+AE^{iv}} & =1.999732275819, & E^{v}B \\
 & AE^{vi} & =\sqrt{2+AE^{v}} & =1.999933067835, & E^{vi}B \\
 & AE^{vii} & =\sqrt{2+AE^{vi}} & =1.999983266889, & E^{vii}B \\
 & AE^{viii} & =\sqrt{2+AE^{vii}} & =1.999995816718, & E^{viii}B \\
 & AE^{ix} & =\sqrt{2+AE^{viii}} & =1.999998954179, & E^{ix}B \\
 & AE^{x} & =\sqrt{2+AE^{ix}} & =1.999999738545, & E^{x}B \\
\text{And} & AE^{xi} & =\sqrt{2+AE^{x}} & -\ \ -\ \ -\ \ - & E^{xi}B
\end{array}\left.\vphantom{\begin{array}{l}1\\1\\1\\1\\1\\1\\1\\1\\1\\1\\1\\1\end{array}}\right\}\text{being the chord of}\left\{\begin{array}{l}
60°; \\
30; \\
15; \\
07.30^{i}; \\
03.45; \\
01.52.30^{ii}; \\
00.56.15; \\
00.28.07.30^{iii}; \\
00.14.03.45; \\
00.07.01.52.30^{iv}; \\
00.03.30.56.15; \\
00.01.45.28.07.30^{v};
\end{array}\right.$$

and this $E^{x\prime}B$ is $(=\sqrt{AB^2-\overline{AE^{xi}}\,^2}\,^{b} = \sqrt{AB^2-2-AE^{x}}) = \sqrt{0.000000261455} = 0.000511326924$: and its half, 0.000255663462 will be the ſine of an arch of $00^{\circ}.\ 00^{i}.\ 52^{ii}.\ 44^{iii}.\ 03^{iv}.\ 45^{v}$ [e]. But ſines of ſmall arches are to one another nearly as the arches themſelves [f]: therefore as the arch $00^{\circ}.\ 00^{i}.\ 52.^{ii}\ 44.^{iii}\ 03.^{iv}\ 45^{v}$. to the arch $00^{\circ}.\ 01^{i}$, ſo is the ſine 0.000255663462 to the ſine 0.000290888205. Wherefore 0.000290888205 is the ſine of an arch of one minute in a circle, whoſe radius is 1.   Q. E. I.

PROPOSI-

[a] El. cor. 15. 4.
[b] El. 31. 3. & 47. 1.
[c] 6. 1.
[d] El. 17. 6.
[d] El. 17. 6.
[b] El. 31. 3. & 47. 1.
[e] 1. cor. 4. def.
[f] 5. 1.

Probl.

PROPOSITION VIII.

Having the radius of a circle, and the ſine of any of its arches given, to find the coſine of that arch.

Figure I.

Let the radius DB, and the ſine BL of an arch AB be given. Therefore in the rightangled triangle BDL, there will alſo be given
a El. 47. 1. $DL = \sqrt{DB^2 - BL^2}$[a], which is equal to the coſine of the arch
b 1. cor. 9. def. AB[b]. Q. E. I.

Probl.

PROPOSITION IX.

To conſtruct the trigonometrical canon.

a 10. def. Although the radius of the circle in the canon is 10,000,000,000[a]; yet, for the conſtruction of it, it is better to take a circle, whoſe radius is 1.

And, to begin with ſines and coſines, the ſine of an arch of one minute,
b 7. 1. in ſuch a circle, is 0.000290888205[b]; whence the coſine alſo of this arch,
c 8. 1. viz. 0.999999957692, will be found[c]. And becauſe Rad. : 2 cos. 1′ ::
d 2 cor. 6. 1. $\overset{\text{Sin.}}{\text{S.}}\ 1' : \overset{\text{Sin.}}{\text{S.}}\ 2'$ :: cos. 1′ : Rad. + cos. 2′[d]; the ſine of an arch of 2′, viz. 0.000581776385, and its coſine, viz. 0.999999830768, will be found. Alſo, Rad. : cos. 1′ :: $\overset{\text{Sin.}}{\text{S.}}\ 2' : \overset{\text{Sin.}}{\text{S.}}\ 1' + \overset{\text{Sin.}}{\text{S.}}\ 3'$ :: cos. 2′ : cos. 1′ + cos. 3′[d]; therefore there is known the ſine of an arch of 3′, viz. 0.000872664515, as alſo its coſine, viz. 0.999999619228. And in the ſame ratio is $\overset{\text{Sin.}}{\text{S.}}\ 3'$ to $\overset{\text{Sin.}}{\text{S.}}\ 2' + \overset{\text{Sin.}}{\text{S.}}\ 4'$ and cos. 3′ to cos. 2′ + cos. 4′; as well as $\overset{\text{Sin.}}{\text{S.}}\ 4'$ to $\overset{\text{Sin.}}{\text{S.}}\ 3' + \overset{\text{Sin.}}{\text{S.}}\ 5'$, and cos. 4′ to cos. 3′ + cos. 5′; and alſo $\overset{\text{Sin.}}{\text{S.}}\ 5'$ to $\overset{\text{Sin.}}{\text{S.}}\ 4' + \overset{\text{Sin.}}{\text{S.}}\ 6'$, and cos. 5′ to cos. 4′ + cos. 6′; and ſo on farther. Wherefore there will be found ſines and coſines of arches of 4′, 5′, 6′, &c. as far as the arch of 30°. And becauſe the firſt term of all theſe analogies is 1, the fourth

fourth term will be found, only by multiplying the third term into the number expreſſing the ſecond term, which always remains the ſame, viz. 2 cos. 1′; which thing may be eaſily performed by the contracted method of multiplication and others, which arithmetick ſhews. Moreover the ſine of the arch of 30° is known[e], as alſo its coſine[c], $= \frac{1}{2}\surd 3$, but beſides that, the ſines and coſines of arches of 3°. 45′, 7°. 30′, 11°. 15′, 15°, 18°. 45′, 22°. 30′, and 26°. 15′, whoſe difference is an arch of 3°. 45′, may be corrected by means of prop. 7. and 1 cor. 4 def. and 2 cor. 9 def. as alſo 2 cor. 6 prop. In this manner will be found ſines and coſines of arches from 1′ to 30°, i. e. ſines of arches from 1′ to 30°, and ſines of arches from 60° to 90°. And the ſines of the remaining arches of the quadrant, i. e. thoſe which are between 30° and 60°, may be more eaſily found by ſubtraction: for $\overset{\text{Sin.}}{\text{S.}}\ 30^\circ.\ 01' = \overset{\text{Sin.}}{\text{S.}}\ 89^\circ.\ 59 - \overset{\text{Sin.}}{\text{S.}}\ 29^\circ.\ 59'$[f]; $\overset{\text{Sin.}}{\text{S.}}\ 30^\circ.\ 02' = \overset{\text{Sin.}}{\text{S.}}\ 89^\circ.\ 58' - \overset{\text{Sin.}}{\text{S.}}\ 29^\circ.\ 58'$[f], &c. $\overset{\text{Sin.}}{\text{S.}}\ 31^\circ = \overset{\text{Sin.}}{\text{S.}}\ 89^\circ - \overset{\text{Sin.}}{\text{S.}}\ 29^\circ$[f], and ſo on, as far as the arch of 60°.

When ſines of all the arches of a quadrant are found, their coſines are known alſo[g].

After ſines and coſines, the tangents and the ſecants of all theſe arches will be found, by prop. 1. and therefore their cotangents and coſecants alſo[g]: the verſed and coverſed ſines alſo will be found[h].

Arches, which are between 90° and 180°, as well as thoſe which are between 180° and 360°, have their ſines, coſines, coverſed ſines, tangents, cotangents, ſecants, and coſecants, the ſame as their ſupplements have[k]: and theſe have been found already: but alſo the verſed ſines of theſe arches may be eaſily found[k].

Of the numbers denoting ſines, coſines, &c. of all arches, logarithms may be taken out of the tables, which then are called artificial, or logarithmical, ſines, coſines, &c. But logarithmical tangents and cotangents, ſecants and coſecants, verſed and coverſed ſines may more eaſily be found by means of logarithmical ſines and coſines alone, by prop. 1. and its cor. and prop. 2.

From ſines, coſines, &c. thus found for arches of a circle, whoſe radius is 1, may very eaſily be derived ſines, coſines, &c. for arches of a circle, whoſe radius is 10,000,000,000[l]: and to the logarithmical ſines, &c. found before, there need only be added 10. And in this manner the trigonometrical canon will be conſtructed[a]. Q. E. F.

SCHO-

[e] 3. 1.
[c] 8. 1.
[f] 4. cor. 6. 1.
[g] 9. def.
[h] 3. cor. 9. def.
[k] 6. cor. 9. def.
[l] 5. cor. 9. def.
[a] 10. def.

SCHOLIUM.

This is the method of conſtruction of the canon of ſines and coſines, which geometry and arithmetick alone afford. The eaſier and more accurate ones depend on higher ſciences. There even the logarithmical ſines and coſines may be immediately found from arches themſelves. It were poſſible indeed, by means of geometry alone, to conſtruct the canon of ſines ſomewhat more correct, if ſines of arches, whoſe difference is 45 minutes, were found: but this demands more principles as well as labour. Tangents alſo and ſecants of many arches might have been found in a ſimpler way, than by means of prop. 1. But it may be preſumed, that with this conſtruction of the canon we may be contented: by the care of induſtrious men we have it done to our hands: it remains for us only to know ſome one method, by which it might be conſtructed, and ſometimes to correct it.

The canon is commonly conſtructed for arches which are leſs than 90°, but the ſame ſerves for greater arches. However coverſed ſines, ſecants, and coſecants are ſeldom in uſe. The logarithmical canon commonly conſiſts of ſeven decimal places only, ſince ſo many ſuffice even for nicer calculations: for accuracy of calculations does not ſo much depend on a great many places in the canon, as on the diviſion of the circumference into a greater number of arches, and the ſetting down of their ſines, &c. in the canon, particularly of thoſe of ſmall arches. For this reaſon, the natural canon alſo is commonly made to conſiſt of ſeven places only, and its radius is 10,000,000; ſo that it correſponds to the logarithmical in the number of places only, and not in the radius.

There is alſo another ſort of trigonometrical canon, natural as well as logarithmical, which is expreſſed not by numbers, but by ſtrait lines: but of this it is treated more at large in books on the conſtruction and uſe of mathematical inſtruments.

Solution of Plane Triangles.

Book I.

LEMMA.

OF two unequal magnitudes, the greater is equal to half the ſum of thoſe magnitudes together with half their difference, and the leſſer is equal to the exceſs of half the ſum above half the difference.

Figure VI.

Let A B, B C be two unequal magnitudes, and A B the greater. Let there be compounded of them a magnitude A C; and of this A C let each of A D, D C be the half; and let D E be equal to D B. Therefore A B is equal to A D, D B together, and B C is the exceſs of D C above D B. And becauſe each of A D, D C is the half of A C, they are therefore equal to one another, and each of them is half the ſum of A B, B C: but D E, D B are alſo equal; wherefore the remaining A E is equal to the remaining B C: and ſo B E is the difference of A B, B C; and D B half that difference. Therefore A B is equal to half the ſum together with half the difference, and B C is equal to the exceſs of half the ſum above half the difference.

Cor. Half the ſum is equal to the exceſs of the greater magnitude above half the difference, and half the difference is equal to the exceſs of the greater magnitude above half the ſum.

PROPOSITION X.

Theor.

In a plane rightangled triangle, as the hypotenuſe to the radius, ſo is either of the two ſides about the right angle, to the ſine of the angle oppoſite to that ſide: and as either of the two ſides about the right angle, to the radius, ſo is the other ſide to the tangent of the angle oppoſite to it, and ſo is the hypotenuſe to the ſecant of the ſame angle.

Figure II.

Let A C G be a plane rightangled triangle, and its angle A right. I ſay, that as the hypotenuſe C G is to the radius, ſo is the ſide A G to the ſine of the angle C, and ſo is the ſide A C to the ſine of the angle G; and as the ſide C A is to the radius, ſo is the other ſide A G to the tangent of the angle C, and ſo the hypotenuſe C G to the ſecant of the ſame angle C.

From the center C, with any radius C D, deſcribe the circumference of a circle, D E, meeting the ſtrait lines C A, C G in the points D, E: from E draw the ſtrait line E H perpendicular to C D, and from D draw D K at right angles to the ſame C D, meeting C G in K.

[a] 4. def.
[b] 6. def.
[c] 8. def.
[d] 1. cor. 9. def.
[e] El. 4. 6.

It is manifeſt, that E H is the ſine [a], D K the tangent [b], and C K the ſecant [c], of the angle C, in the circle D E; and C H is the coſine of the angle C [d], i. e. the ſine of the angle E, or G, in the ſame circle D E. But in the equiangular triangles A C G, H C E, as C G to C E, ſo is G A to E H [e], and C A to C H [e]; and in the equiangular triangles A C G, D C K, as A C to C D, ſo is A G to D K [e], and G C to C K [e]. Wherefore the propoſition is manifeſt.

Theor.

PROPOSITION XI.

In a plane rightangled triangle, as the hypotenuſe together with one of the two ſides about the right angle, is to the radius, ſo is the other ſide to the tangent of half the angle oppoſite to it.

Figure I.

Let B D L be a plane rightangled triangle, and its angle L right. I ſay, that as B D, D L together to the radius, ſo is B L to the tangent of half the angle B D L.

From the center D, at the diſtance D B, deſcribe the circumference of a circle, C B A, meeting D L produced in the points C, A; and join C B.

[a] El. 20. 3.
[b] 10. 1.

Therefore the angle B C L is half the angle B D L [a]. And becauſe the triangle B C L is rightangled, as C L, i. e. as B D, D L together, to the radius, ſo is B L to the tangent of the angle B C L [b], i. e. to the tangent of half the angle B D L.

[c] El. 31. 3. & cor. 8. 6.
[d] El. 2. cor. 20. 6.

Cor. Join B A. And it will be as C L to B L, ſo B L to A L [c]: therefore as C L to A L, ſo is the ſquare of C L to the ſquare of B L [d]; i. e. as the ſum of B D, D L together, to their difference, ſo is the ſquare of B D, D L together to the ſquare of B L. Wherefore, if there be taken numbers, to expreſs the ſum and the difference of the ſtrait lines B D, D L; it will be, as $\sqrt{BD + DL}$ to $\sqrt{BD - DL}$, ſo B D + D L to B L, and ſo the radius to the tangent of half the angle B D L, by the prop.

PROPOSITION XII.

Theor.

In every plane triangle, the ſides are to one another, as the ſines of their oppoſite angles.

In rightangled triangles this is manifeſt: for as the hypotenuſe is to the radius, i. e. to the ſine of the right angle which is oppoſite[a], ſo is either of the two ſides to the ſine of the angle oppoſite to it[b].

Fig. VII. VIII.

But let the triangle A B C be obliqueangled. I ſay, that as B A to A C, ſo is the ſine of the angle C to the ſine of the angle B.

From A, the point of meeting of the ſides B A, A C, draw the ſtrait line A D perpendicular to the baſe B C, produced, if neceſſary. And becauſe the triangle A B D is rightangled, as B A to the radius, ſo is A D to the ſine of the angle B[b]. For the ſame reaſon, as the ſame radius is to A C, ſo is the ſine of the angle A C D, or A C B[c], to A D: therefore, ex æquali perturbatè, as B A to A C, ſo is the ſine of the angle C to the ſine of the angle B[d]. Q. E. D.

Cor. In like manner, from the ſecond part of prop. 10. it will be demonſtrated, that the ſegments of the baſe B D, D C, which lay between the ends of the baſe and the perpendicular A D, are to one another, as the tangents of the angles at the vertex, B A D, D A C.

PROPOSITION XIII.

Theor.

Figure IX.

In every plane triangle, A B C, as the ſum of two unequal ſides B A, A C, to their difference, ſo is the cotangent of half the angle A, contained by thoſe ſides, or the tangent of half the ſum of the angles at the baſe, A C B,

[a] 2. cor. 4. def.

[b] 10. 1.

[c] 6. cor. 9. def.

[d] El. 23. 5.

CBA, to the tangent of half the difference of thoſe angles ACB, CBA.

From AB take away AD equal to AC, and join DC; and biſect the angle BAC by the ſtrait line AEF; and through E draw EG parallel to BC.

Therefore in the triangles AEC, AED, CE is equal to ED[a], the angle ACE is equal to the angle ADE, and the angles at E are right ones[a]. And becauſe CE is equal to ED; but CE is to ED, as BG to GD, becauſe GE is parallel to BC[b]: therefore BG alſo is equal to GD[c]; ſo that BG is half of BD. And BD is the difference of the ſides BA, AC: wherefore BG is half their difference. But of two unequal magnitudes, the exceſs of the greater above half the difference is equal to half the ſum of thoſe magnitudes[d]: therefore GA is half the ſum of the ſides BA, AC. And becauſe the angle BAC is common to the two triangles ADC, ABC; the remaining angles ACD, ADC together, (i. e. twice the angle ACE, or ADE) are equal to the remaining angles ACB, CBA together[e]: and ſo the angle ACE, or ADE, is equal to half the ſum of the angles ACB, CBA: and the exceſs of the angle ACB, the greater of thoſe angles, above half the ſum ACE, i. e. the angle ECF, is equal to half the difference of the angles ACB, CBA[d].

[a] El. 4. 1.
[b] El. 2. 6.
[c] El. A. 5.
[d] cor. lem.
[e] El. 32. 1.
[d] cor. lem.

And becauſe EG is parallel to BF; as AG to GB; ſo is AE to EF[b]. But as AE to EF, ſo is the tangent of the angle ACE, or the cotangent of the angle CAE, to the tangent of the angle ECF[f]: therefore as AG to GB, or as twice AG to twice GB[g], ſo is the tangent of the angle ACE, or the cotangent of the angle CAE, to the tangent of the angle ECF: i. e. as the ſum of the ſides BA, AC, to their difference, ſo is the cotangent of half the contained angle A, or the tangent of half the ſum of the angles at the baſe, ACB, CBA, to the tangent of half the difference of thoſe angles. Q. E. D.

[b] El. 2. 6.
[f] cor. 12. 1.
[g] El. 15. 5.

Figure IX. X.

Cor. 1. In every plane triangle ABC, the leſſer ſide CA : greater AB :: Rad. : Tang. of an angle; and Rad : tang. of $\overline{\text{this angle} - 45^\circ}$ ::

$$\text{tang.}\ \frac{BCA+B}{2}\ (= \text{cotang.}\ \tfrac{1}{2}A) : \text{tang.}\ \frac{BCA-B}{2}.$$

Take the triangle NOP, whoſe angle ONP is right, and ON : NP :: BA : AC. make NR = NP, and join RP. Therefore

1ſt. CA : AB :: (PN : NO ::) Rad. : tang. NPO[h], which is greater than 45°[k].

2dly. Becauſe ON : NP :: BA : AC; componendo & dividendo, ON + NP : ON − NP :: BA + AC : BA − AC (:: tang. $\frac{\text{NPO} + \text{O}}{2}$: tang. $\frac{\text{NPO} - \text{O}}{2}$[l] :: tang. NPR or 45° : tang. RPO, or tang. $\overline{\text{NPO} - \text{NPR}}$) :: Rad.[m] : tang. $\overline{\text{NPO} - 45°}$ :: tang. $\frac{\text{BCA} + \text{B}}{2}$ (= cot. $\frac{1}{2}$ A) tang. $\frac{\text{BCA} - \text{B}}{2}$[l].

In like manner it may alſo be demonſtrated, that the $\underline{\text{greater ſide BA}}$: leſſer AC :: Rad. : tang. of an angle; and Rad. : tang. $\overline{45° - \text{this angle}}$[k] :: tang. $\frac{\text{BCA} + \text{B}}{2}$ (= cot. $\frac{1}{2}$ A) : tang. $\frac{\text{BCA} - \text{B}}{2}$.

Cor. 2. Sin. $\frac{\text{BCA} + \text{B}}{2}$ (= cos. $\frac{1}{2}$ A) : Sin. $\frac{\text{BCA} - \text{B}}{2}$ :: BC : BA − AC; and cos. $\frac{\text{BCA} + \text{B}}{2}$ (= $\overset{\text{Sin.}}{\text{S.}}$ $\frac{1}{2}$ A) : cos. $\frac{\text{BCA} - \text{B}}{2}$ :: BC : BA + AC.

Figure IX.

For 1ſt. In the triangle BCD, $\overset{\text{Sin.}}{\text{S.}}$ BDC (= $\overset{\text{Sin.}}{\text{S.}}$ ADE[n] = cos. $\frac{1}{2}$ A) : $\overset{\text{Sin.}}{\text{S.}}$ BCD :: BC : DB[o] = BA − AC.

2dly. In the triangle AEG, $\overset{\text{Sin.}}{\text{S.}}$ GAE : $\overset{\text{Sin.}}{\text{S.}}$ AEG :: GE : AG[o]. But the angle GAE is the complement of the angle ADE, or of $\frac{\text{BCA} + \text{B}}{2}$; and the ſine of the angle AEG is the ſame with that of the angle GEF[n], which is the complement of the angle GED, or BCD[p], i. e. of $\frac{\text{BCA} - \text{B}}{2}$; and GE is the half of BC[q]: therefore cos. $\frac{\text{BCA} + \text{B}}{2}$ (= $\overset{\text{Sin.}}{\text{S.}}$ $\frac{1}{2}$ A) : cos. $\frac{\text{BCA} - \text{B}}{2}$ :: $\frac{1}{2}$ BC : $\frac{\text{BA} + \text{AC}}{2}$ :: BC : BA + AC[s].

PROPOSI-

[h] 10. 1.

[k] cor. 4. 1. & El. A. 5.

[l] prop.

[m] 4. 1.

[n] 6 cor. 9. def.

[o] 12. 1.

[p] El. 29. 1.

[q] El. 4. 6. & D. 5.

[s] El. 15. 5.

Theor.

PROPOSITION XIV.

Figure IX.

In every plane triangle A B C, as the rectangle contained by half the perimeter of the triangle, and the exceſs of it above the baſe B C, to the rectangle contained by the exceſſes of half the perimeter above each of the two remaining ſides, B A, A C; ſo is the ſquare of the radius to the ſquare of the tangent of half the angle B A C contained by thoſe two ſides.

Make the ſame conſtruction, as in the preceding prop. and alſo from the center G, at the diſtance G E, deſcribe the circumference of a circle H K L, meeting E G, A B produced in the points H, K; join H B, and produce it, till it meet A E in M.

[a] El. 4. 1.
[b] El. 4. 6.
[c] El. 29. 1.
[d] El. D. 5.
[e] El. 33. 1.
[a] El. 4. 1.
[c] El. 29. 1.

Becauſe D E is half D C[a]; and as D E to D C, ſo is D G to D B[b], and G E to B C[b]; for the triangles D E G, D C B are equiangular[c]: therefore D G alſo is half D B[d], and G E half B C[d]: wherefore the whole E H is equal to the whole B C. But E H is alſo parallel to B C: therefore C E alſo, or E D, is equal and parallel to H B[e]. And becauſe E D is parallel to H M, and that the angle A E D is right[a]; the angle E M H is right alſo[c], ſo that the point M muſt be in the circumference of the circle H K L, as appears from El. 31. 3.

[f] or. lem.

And becauſe A G is equal to half the ſum of the ſides B A, A C[f]; add G K, or G E, which is half B C: and A K will be equal to half the perimeter of the triangle A B C. From A K take away K L, or H E, i. e. B C: and A L will be equal to the exceſs of half the perimeter A K above the baſe B C. And B K is the exceſs of half the perimeter A K above A B. And becauſe G B is equal to G D, and G L to G K: wherefore the whole B L is equal to the whole D K. But D K is the exceſs of half the perimeter A K above A D, or A C: therefore B L alſo is the exceſs of half the perimeter A K above A C.

10. 1.

And becauſe as A M to M B, ſo is the radius to the tangent of the angle D A E[g]; and as A E to E D, ſo is the radius to the tangent of the angle D A E[g]: as therefore the rectangle M A, A E, to the rectangle M B, D E, D E, or to the rectangle M B, B H, ſo is the ſquare of the radius to the ſquare of the tangent of the angle D A E[h]. But the rectangle M A, A E is equal to the rectangle K A, A L[k]; and the rectangle M B, B H is equal to the rectangle K B, B L[l]: therefore, as the rectangle K A, A L, (contained by half the perimeter of the triangle, and the exceſs of half the perimeter above the baſe B C), to the rectangle K B, B L, (contained by the exceſſes of half the perimeter above each of the ſides B A, A C); ſo is the ſquare of the radius to the ſquare of the tangent of the angle D A E, i. e. to the ſquare of the tangent of half the angle B A C, by the conſtruction. Q. E. D.

COR. 1. Put half the perimeter of the triangle A B C, $= \frac{1}{2}P$; and it will be $\overline{\frac{1}{2}P - AB} \times \overline{\frac{1}{2}P - AC} : \frac{1}{2}P \times \overline{\frac{1}{2}P - BC} :: \text{Rad}^2.$ cotangent^2 $\frac{1}{2}A$; which may be demonſtrated after a like manner, or by means of the cor. prop. 1.

COR. 2. $AB \times AC : \overline{\frac{1}{2}P - AB} \times \overline{\frac{1}{2}P - AC} :: \text{Rad}^2. : \text{Sin}^2. \frac{1}{2}A.$ alſo, $AB \times AC : \frac{1}{2}P \times \overline{\frac{1}{2}P - BC} :: \text{Rad}^2. : \cos^2. \frac{1}{2}A.$

For 1ſt. $AB : BM :: \text{Rad.} : \text{Sin.} \frac{1}{2}A$[g]; and $AD : DE :: \text{Rad.} : \text{Sin.} \frac{1}{2}A$[g]; wherefore $(AB \times AD =) AB \times AC : (BM \times DE =) KB \times BL :: \text{Rad}^2. : \text{Sin}^2. \frac{1}{2}A$[h].

2dly. $AB : AM :: \text{Rad.} : \cos. \frac{1}{2}A$[g]; and $AD : AE :: \text{Rad.} : \cos. \frac{1}{2}A$[g]; wherefore $(AB \times AD =) AB \times AC : (AM \times AE =) KA \times AL :: \text{Rad}^2. : \cos^2. \frac{1}{2}A$[h]. And the rectangle $\overline{\frac{1}{2}P - AB} \times \overline{\frac{1}{2}P - AC} = \frac{1}{4} \times \overline{BC + AC - AB} \times \overline{BC + AB - AC}$; and the rectangle $\frac{1}{2}P \times \overline{\frac{1}{2}P - BC} = \frac{1}{4} \times \overline{AB + AC + BC} \times \overline{AB + AC - BC}$, as may appear from the common notions.

COR. 3. $AB \times AC : \overline{\frac{1}{2}P - AB} \times \overline{\frac{1}{2}P - AC} :: 2\,\text{Rad.} : \text{vers. ſin. } A.$ alſo, $AB \times AC : \frac{1}{2}P \times \overline{\frac{1}{2}P - BC} :: 2\,\text{Rad.} : \text{vers. ſin. ſuppl}^{r}\ A.$

For 1ſt. Becauſe $\frac{1}{2}\text{Rad.} : \text{ſin.} \frac{1}{2}A : \text{vers. ſin. } A$[k]; therefore $\frac{1}{4}\text{Rad}^2. : \text{Sin}^2. \frac{1}{2}A :: \frac{1}{2}\text{Rad.} : \text{vers. ſin. } A$[l], and $\text{Rad}^2. : \text{ſin}^2. \frac{1}{2}A :: 2\,\text{Rad.} : \text{vers. ſin. } A$[m]. Wherefore $AB \times AC : \overline{\frac{1}{2}P - AB} \times \overline{\frac{1}{2}P - AC} :: (\text{Rad}^2. : \text{ſin}^2. \frac{1}{2}A$[n] $::)\ 2\,\text{Rad.} : \text{vers. ſin. } A.$ And this analogy may be ſplit into two, thus. $AC : \frac{1}{2}P - AB :: \frac{1}{2}P - AC$: a ſtrait line, and AB : that ſtrait line $:: 2\,\text{Rad.} : \text{vers. ſin. } A$[o]. After a like manner the ſecond analogy may be demonſtrated and ſplit into two.

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[h] El. 23. 6.

[k] El. cor. 36. 3.

[l] El. 35. 3.

[g] 10. 1.

[h] El. 23. 6.

[k] 2. 1.

[l] El. 2. cor. 20. 6.

[m] El. cor. 4. 5.

[n] 2. cor.

[o] El. F. 5. & 1. 6.

Cor. 4. From the prop. and the Elements it may be demonſtrated, that $\frac{1}{2}P - AC : \frac{1}{2}P - BC :: \text{tang. } \frac{1}{2}A : \text{tang. } \frac{1}{2}B$.

Cor. 5. $\frac{1}{2}P \times \overline{\frac{1}{2}P - BC} : \text{triang. } ABC : \overline{\frac{1}{2}P - AB} \times \overline{\frac{1}{2}P - AC}$.

[p] El. 41. 1.
[q] El. 1. 2.
[b] El. 4. 6.
[r] El. 1. 6.

For the triang. $DCA = DE \times EA$[p], and the triang. $DCB = DE \times EM$[p]; wherefore the whole triang. $ABC = DE \times AM$[q]. But becauſe $AE : ED :: AM : MB$[b]; it will therefore be $(MA \times AE =) KA \times AL : (AM \times DE$[r] $=)$ triang. $ABC : : (AM \times DE =)$ triang. $ABC : (MB \times DE$[r] $=) KB \times BL$.

Theor.

PROPOSITION XV.

Fig. VII. VIII.

In every plane triangle ABC, the ſum and the difference of two unequal ſides BA, AC are reciprocally proportional to the ſum and the difference of the ſegments of the baſe, BD, DC, which lay between the ends of the baſe, B, C, and the perpendicular AD drawn upon the baſe from the vertical angle BAC.

From the center A, at the diſtance of the greater ſide AB, deſcribe the circumference of a circle, meeting AC, BC produced in the points E, F, G.

[a] El. 3. 3.
[b] El. 35. 3.
[c] El. 16. 6.

It is manifeſt, that FC is the ſum of the ſides BA, AC; and CE their difference. And becauſe AD is perpendicular to BG; BD is equal to DG[a]: wherefore of the two BC, CG, one is the ſum of the ſegments of the baſe, DB, DC; and the other their difference. But the rectangle FC, CE is equal to the rectangle BC, CG[b]: wherefore FC, CE, the ſum and the difference of the ſides BA, AC, are reciprocally proportional to BC, CG, the ſum and the difference of the ſegments of the baſe, DB, DC[c]. Q. E. D.

SCHOLIUM

SCHOLIUM

To the following Tables of Solutions of Plane Triangles.

In every plane triangle, if of the ſix of its elements, which are the three ſides and the three angles, any three be given, the remaining three alſo will be found from the foregoing principles; except when the three angles are given, in which caſe the ratio of the ſides only will be found, by prop. 12. and one of them may then be aſſumed at pleaſure. For which reaſon this caſe commonly is not reckoned amongſt the others.

In every plane triangle, if two angles be given, the third alſo is given; and if one angle be given, the ſum of the other two is given, El. 32. 1.

A plane rightangled triangle, which has the ſides about the right angle equal, or which has one of the oblique angles equal to half a right angle, i. e. of 45°, needs no ſolution: for the hypotenuſe is to either of the ſides, as $\surd 2 = 1.414213562373$ to 1. El. 47. 1.

Alſo, if in a plane rightangled triangle, one of the angles be of 30°, then the ſide oppoſite to it is half of the hypotenuſe; and the hypotenuſe is to the remaining ſide, as 1 to $\frac{1}{2}\surd 3 = 0.866025403784$. And converſely, if in a plane rightangled triangle, the hypotenuſe be double of one of the ſides, the angle oppoſite to that ſide is of 30°, &c. prop. 10. and 3. & El. 47. 1.

Of plane obliqueangled triangles, an equilateral triangle, or that, whoſe two angles are, each of them, equal to a third of two right angles, i. e. each of 60°, needs no ſolution.

Alſo, of plane obliqueangled triangles, thoſe that have two ſides, or two angles, equal, may be more eaſily ſolved, if from the vertex of the third angle, upon the third ſide, there be drawn a perpendicular, which will biſect both that angle and that ſide. El. 47. 1. Wherefore it will be ſufficient to ſolve one of the rightangled triangles, which are on each ſide of the perpendicular.

Book I.

In plane triangles, rightangled as well as obliqueangled, there is no double ſolution, excepting the firſt caſe of the obliqueangled; which correſponds ſomewhat to El. 7. 6. wherein two triangles, having the ſame data, may have their remaining elements unequal. All the other caſes have ſingle ſolutions, as appears from El. Book I. prop. 4, 8, 26, & 47.

Figure XI.

TABLE OF SOLUTIONS

Of Plane Rightangled Triangles.

	Given.	Sought.	The Solution.
1.	BC, CA; hypotenuſe and a ſide.	the angles B, C. the ſide AB.	$BC : \text{Rad.} :: AC : \text{Sin. } B = \text{Cos. } C$. prop. 10. Find firſt the angle C, or B. Then Rad. : BC :: Sin. C (= cos. B) : AB. prop. 10. Otherwiſe, without finding of the angles B, C. $\sqrt{\overline{BC + CA} \times \overline{BC - CA}} = AB$. El. 47. 1. & 6. 2. Or both the quaeſita together. $\sqrt{BC + CA} : \sqrt{BC - CA} ::$ Rad. : tang. $\frac{1}{2}$ C :: $BC + CA : AB$. cor. prop. 11.
2.	BC, B; hypotenuſe and an angle.	the ſide oppoſite to the angle, AC. the ſide adjacent to the angle, AB.	$\text{Rad.} : BC :: \left\{\begin{array}{l}\text{Sin. } B \quad : \quad AC \\ \text{Cos. } B\,(= \text{Sin. } C) : AB\end{array}\right\}$ prop. 10.
3.	BA, AC; two ſides.	the angles B, C. the hypotenuſe BC.	BA : Rad. :: AC : tang. B. prop. 10. AC : Rad. :: AB : tang. C. prop. 10. Find firſt the angle B, or C. Then Sin. B : AC :: (Sin. C : AB ::) Rad. : BC. prop. 10. Otherwiſe, without finding of the angles B, C. $\sqrt{BA^2 + AC^2} = BC$. El. 47. 1.
4.	AC, B; a ſide and the angle oppoſite.	the other ſide AB. the hypotenuſe BC.	Tan. B : AC :: Rad. : AB. prop. 10. Sin. B : AC :: Rad. : BC. prop. 10. Otherwiſe. Tang. $\frac{1}{2}$ B : Rad. :: AC : BC + BA, and Rad. : tang. $\frac{1}{2}$ B :: $\sqrt{BC + BA} : \sqrt{BC - BA}$. cor. prop. 11. whence BC − BA is given; and therefore BC, BA will be found, by the lemma.
5.	AB, B; a ſide and the angle adjacent.	the other ſide AC, and the hypotenuſe BC.	This caſe is the ſame with the preceding; for the angle oppoſite to the given ſide is known.

Book I.

Fig. VII. VIII.

TABLE OF SOLUTIONS

Of Plane Obliqueangled Triangles.

	Given.	Sought.	The Solution.
1.	BA, AC, and B; two ſides, and one of the oppoſite angles.	the other of the oppoſite angles, C.	AC : Sin. B :: AB : Sin. C, or to the ſine of its ſupplement, prop. 12. The ſolution is double, except when the angle B and the ſupplement of the angle C are together not leſs than two right angles, in which caſe the ſupplement of the angle C cannot have place. El. 17. 1.
		the third angle, A.	Find firſt the angle C, and then the angle A will alſo be known, and if the ſolution in regard to the angle C is double, it will be double in regard to the angle A alſo. And when the ſolution is double, if the angle C be acute, then the angle A is the ſupplement of the angles B, C; but if the angle C be obtuſe, the angle A is then the difference of the acute angle C and of the angle B.
		the third ſide, BC.	Find the angle A. Then Sin. B : AC :: Sin. A : BC. prop. 12. and if the ſolution in regard to the angle A is double, it will be double in regard to BC alſo.
2.	B, C, and AB; two angles, and one of the oppoſite ſides.	the remaining ſides, AC, BC.	Becauſe the angles B, C are given, there is given alſo the third angle A. Therefore $\text{Sin. C : AB} :: \left\{\begin{matrix}\text{Sin. B : AC}\\ \text{Sin. A : BC}\end{matrix}\right\}$ prop. 12.

Becauſe

Fig. VII. VIII.

	Given.	Sought.	The Solution.
3.	BA, AC, and A; two ſides and the contained angle.	the angles C, B.	Becauſe the angle A is given, there is given the ſum of the remaining two angles. Therefore $BA + AC : BA - AC ::$ tang. $\frac{C + B}{2}$ (= cot. $\frac{1}{2}$ A) : tang. $\frac{C - B.}{2}$ prop. 13. and hence each of the angles C, B will be known, by the lemma. Otherwiſe, $\left\{\begin{matrix} CA : AB \\ BA : AC \end{matrix}\right\}$ :: Rad. : tang. of an angle; and Rad. : tang. of $\left\{\begin{matrix} \overline{\text{this angle} - 45^\circ} \\ \hline 45^\circ - \text{this angle} \end{matrix}\right\}$ :: tang. $\frac{C + B}{2}$ (= cot. $\frac{1}{2}$ A) : tang. $\frac{C - B.}{2}$ 1. cor. 13. prop. whence the angles C, B will be found, as before.
		the third ſide, BC.	Find firſt the angles C, B. Then Sin. $\frac{C - B}{2}$: Sin. $\frac{C + B}{2}$ (= cos. $\frac{1}{2}$ A) :: $BA - AC : BC$. 2. cor. 13. prop. Or Cos. $\frac{C - B}{2}$: cos. $\frac{C + B}{2}$ (= ſin. $\frac{1}{2}$ A) :: $BA + AC : BC$. 2. cor. 13. prop. Or again Sin. C : AB :: (ſin. B : AC ::) ſin. A : BC. prop. 12. Otherwiſe, without finding of the angles B, C. Rad.^{2} : Sin.^{2} $\frac{1}{2}$ A :: $4\,AB \times AC : BC^2 - \overline{BA - AC}^2$. 2. cor. 14. prop. & El. 6. 2. whence, by adding $\overline{BA - AC}^2$, there is found BC^2, and therefore BC. Or Rad.^{2} : cos.^{2} $\frac{1}{2}$ A :: $4\,AB \times AC : \overline{AB + AC}^2 - BC^2$. 2. cor. 14. prop. & El. 6. 2. and by ſubtracting this from

Book I.

Fig. VII.
VIII.

	Given.	Sought.	The Solution.
			from $\overline{AB+AC}^2$, there will be found BC^2, and therefore BC, as appears from the common notions.
4.	B, C, and BC; two angles and the ſide adjacent to them.	the remaining ſides, BA, AC.	This caſe is the ſame with the ſecond, becauſe the angle A, oppoſite to the given ſide BC, is known alſo. But it may alſo be ſolved thus. Sin. $\frac{C+B}{2}$: ſin. $\frac{C-B}{2}$ :: BC : BA − AC, and Cos. $\frac{C+B}{2}$: cos. $\frac{C-B}{2}$ :: BC : BA + AC. 2. cor. 13. prop. whence BA, AC themſelves will be known, by the lemma.
5.	AB, AC, BC; the three ſides.	the angle A.	Put half the perimeter of the triangle, $\frac{AB+AC+BC}{2} = \frac{1}{2}P$. and it will be $\frac{1}{2}P \times \overline{\frac{1}{2}P - BC} : \overline{\frac{1}{2}P - AB} \times \overline{\frac{1}{2}P - AC}$ :: Rad.^{2} : tang.^{2}. ½ A. prop. 14.
		the angle B.	$\frac{1}{2}P \times \overline{\frac{1}{2}P - AC} : \overline{\frac{1}{2}P - AB} \times \overline{\frac{1}{2}P - BC}$ :: Rad.^{2} : tang.^{2} ½ B. prop. 14. Or the angle B will be found from the angle A, thus. ½ P − AC : ½ P − BC :: tang. ½ A : tang. ½ B. 4. cor. 14. prop. After this, the third angle C will be known alſo. Otherwiſe. Let either of the angles B, C, be to be found. To one of the ſides of the angles ſought, BC, as to the baſe, draw, from the vertex A, the perpendicular AD. And it will be ½ BC : $\frac{BA+AC}{2}$ :: $\frac{BA-AC}{2}$: a ſtrait line, prop. 15. The ſum of this ſtrait line and of

Fig. VII. VIII.

Given.	Sought.	The Solution.
		of $\frac{1}{2}$ B C will be equal to the greater ſegment of the baſe B C, and their difference to the leſſer, by the lemma. And the greater ſegment of the baſe is always adjacent to the greater of the ſides B A, A C, and the leſſer to the leſſer. El. 47. 1. And if the greater ſegment be equal to B C, the angle oppoſite to the greater of the ſides B A, A C, will be a right one; if it be leſſer than B C, the angle will be acute; and if greater, obtuſe: but the angle oppoſite to the leſſer ſide is always acute. Afterwards the angles B, C, or the ſupplement of one of them, will be found by means of the rightangled triangles A B D, A C D.

ELEMENTS
OF
TRIGONOMETRY,
BOTH
PLANE AND SPHERICAL.

BOOK II.

Spherical Trigonometry.

DEFINITIONS.

1. A Circle of the ſphere is that whoſe circumference is upon the ſurface of the ſphere.

2. The axis of a circle of the ſphere is called a ſtrait line drawn from its center at right angles to its plane, and terminated both ways by the ſurface of the ſphere.

3. And the ends of the axis are called the poles of that circle.

Corollary 1. A ſtrait line, paſſing through the poles of any circle of the ſphere, paſſes through its center alſo, and is at right angles to its plane.

For it coincides with the axis of that circle [a].

Cor. 2. A ſtrait line, paſſing through one of the poles and the center of any circle of the ſphere, paſſes through the other pole alſo, and is at right angles to the plane of that circle.

[a] El. 10. ax.

For this alſo coincides with its axis[a].

[a] El. 10. ax.

Cor. 3. A ſtrait line, drawn from one of the poles of any circle of the ſphere, perpendicularly to the plane of it, paſſes through the center alſo and through the other pole of that circle.

[b] El. 13. 11.

For this ſtrait line alſo coincides with the axis of that circle[b].

Cor. 4. All ſtrait lines, as alſo all arches of equal circles, drawn from the pole of any circle of the ſphere to the circumference of that circle, are equal to one another[c].

[c] 3 cor & El. 47. 1.

Cor. 5. If from any point upon the ſurface of the ſphere there be drawn to the circumference of any one of its circles, more than two equal ſtrait lines, or more than two equal arches of equal circles; that point is the pole of that circle[d].

[d] El. 47. 1. & 9. 3.

4. A great circle of the ſphere is called that, whoſe center is the ſame with that of the ſphere.

Cor. 1. Circumferences of great circles of the ſphere cut one another in two points, and into two equal parts.

[a] def.

[b] El. 3. 11.

For their planes paſs through the center of the ſphere[a]: therefore thoſe planes cut one another, and their common ſection is a ſtrait line paſſing through the center of the ſphere[b]. But the center of the ſphere is the ſame with thoſe of theſe circles[a]: therefore that ſtrait line is the common diameter of thoſe circles. And a diameter of a circle cuts the circumference of it in two points, and into two equal parts, as appears from the Elements: therefore circumferences of great circles of the ſphere cut one another in two points, and into two equal parts.

Cor. 2. Great circles of the ſphere cannot have the ſame pole.

For, if they have, the ſtrait line, drawn from their common pole to their common center[a], will be at right angles to the plane of each of them[c]: wherefore thoſe circles will be in one plane[d]; which is abſurd[e].

[c] 2. cor. 3. def.

[d] El. 5. 11.

[e] 1. cor.

5. A leſſer circle of the ſphere is called that whoſe center is not the ſame with that of the ſphere.

6. Parallel circles of the ſphere are ſuch, whoſe planes are parallel.

7. A ſpheri-

7. A ſpherical angle is called the mutual inclination of the planes of great circles of the ſphere; but commonly that angle alſo is called a ſpherical angle, which is contained by arches of great circles.

8. If the angle of the inclination of the planes of the circles be a right one, the ſpherical angle is called right; if not, it is called oblique.

9. A ſpherical triangle is called a figure contained by three arches of great circles of the ſphere, each of which arches is leſſer than a ſemicircumference.

10. A rightangled ſpherical triangle is called that which has a right angle.

11. A quadrantal ſpherical triangle, which has a ſide equal to a quadrant.

12. An obliqueangled ſpherical triangle, which has neither a right angle, nor a ſide equal to a quadrant.

Book II.

Of Circles of the Sphere.

Theorem.

PROPOSITION I.

IF the ſurface of the ſphere be cut by a plane, their common ſection is the circumference of a circle; and this circle will be the greateſt of all, when the cutting plane paſſes through the center of the ſphere.

Plate II.
Figure I.

Let their common ſection be A B C, which is a line [a].

If the plane paſſes through the center of the ſphere, the line A B C is the circumference of a circle, whoſe center is the ſame with that of the ſphere, as it is demonſtrated in El. 17. 12.

But if the plane does not paſs through the center of the ſphere, which let be the point D; from D draw the ſtrait line D E perpendicular to the cutting plane, meeting it in the point E; and from E to the line A B C draw any ſtrait lines E A, E C; and join D A, D C.

Becauſe D E is perpendicular to the plane A B C, it is perpendicular to every ſtrait line meeting it in that plane [b]. But in that plane the ſtrait lines A E, E C meet it: therefore each of the angles A E D, D E C is right. And becauſe A D is equal to D C, as appears from El. 14. & 16. def. 11. the ſquare of A D is equal to the ſquare of D C. But the ſquare of A D is equal to the ſquares of A E, E D [c]; and the ſquares of D C to thoſe of C E, E D [c], becauſe the angles at E are right: therefore the ſquares of A E, E D are equal to the ſquares of C E, E D. take away the common ſquare of E D: and the remaining ſquares of E A, E C will be equal; and therefore the ſtrait lines E A, E C equal. In the ſame manner it may be demonſtrated, that all ſtrait lines, drawn from the point E to the line A B C, are equal to one another. And the point E is within the line A B C, becauſe D E is leſſer than D A [d]: Therefore the line A B C is the circumference of a circle, whoſe center is the point E.

[a] El. 6. def. 1.

[b] El. 3. def. 11.

[c] El. 47. 1.

[d] El. 17. & 19. 1.

And becauſe A D is greater than A E[d], the ſquare of A D is greater than the ſquare of A E. But as the ſquares of A D, A E, ſo are the circles, of which A D, A E are the radii[e]: therefore the circle from the radius A D is greater than that from the radius A E[f]. But the circle from the radius A D is equal to every circle paſſing through D the center of the ſphere; and the circle from the radius A E is ſome other circle, not paſſing through the center of the ſphere: therefore a circle paſſing through the center of the ſphere is the greateſt of all. Q. E. D.

COR. 1. Hence it appears, how through two given points upon the ſurface of the ſphere to draw a great circle of the ſphere, or through three given points to draw ſome circle of the ſphere.

COR. 2. A ſtrait line, drawn from the center of the ſphere perpendicularly to the plane of any one of its circles, falls upon the center of that circle; and is thereſore the axis of it[g].

COR. 3. A ſtrait line, drawn from the center of the ſphere to the center of any one of its circles, is perpendicular to the plane of that circle, and is the axis of it.

For if it be not, draw from the center of the ſphere a ſtrait line perpendicular to the plane of the circle. It will fall upon a point, which is alſo the center of that circle[h]; ſo that a circle has two centers, which is impoſſible.

COR. 4. The axis of every circle of the ſphere paſſes through the center of the ſphere.

In great circles it is manifeſt[k]. And in leſſer circles, from the center of the ſphere draw to the center of the circle a ſtrait line. And it will be perpendicular to the plane of the circle[l]: but the axis alſo is perpendicular to the ſame plane at the ſame point[g]: wherefore the axis coincides with the ſtrait line, which is drawn[m]; and therefore it paſſes through the center of the ſphere.

COR. 5. A ſtrait line, drawn from the pole of any circle through the center of the ſphere, is the axis of that circle.

For it coincides with the axis[n].

PROPO-

[d] El. 17. & 19. 1.

[e] El. 2. 12. & 15. 5.

[f] El. A. 5.

[g] 2. def. 2.

[h] 2. cor.

[k] 4. def. 2.

[l] 3. cor.

[g] 2. def. 2.

[m] El. 13 11.

[n] 4. cor.

Theor.

PROPOSITION II.

Great circles of the ſphere, perpendicular to ſome other circle of the ſphere, paſs through the pole of that circle; and great circles of the ſphere, paſſing through the pole of another circle, are perpendicular to that circle.

For, firſt, if from the center of the ſphere there be drawn a ſtrait line perpendicular to that other circle, this ſtrait line will be the axis of that circle[a], and therefore will paſs through its pole. But great circles of the ſphere, perpendicular to that other circle, do paſs through this ſtrait line[b], becauſe the center of the ſphere is in it[c]: wherefore they will alſo paſs through the pole of that circle.

[a] 2. cor. 1. 2.
[b] El. 38. 11.
[c] 4. def. 2.

Secondly, great circles of the ſphere, paſſing through the pole of ſome other circle of the ſphere, paſs through a ſtrait line, which is drawn from that pole through the center of the ſphere[c], i. e. they paſs through the axis of that circle[d], and are therefore perpendicular to that circle[e].

[d] 5. cor. 1. 2.
[e] 2. def. 2. & El. 18. 11.

Cor. 1. If a great circle of the ſphere paſs through the pole of another great circle of the ſphere; converſely, this other great circle will paſs through the pole of the firſt[f].

[f] prop.

Cor. 2. The arch of a great circle between the pole and circumference of another great circle, is a quadrant.

For if a ſtrait line be drawn from the pole to the center of the ſphere, i. e. to that of the arch[c], it will be perpendicular to the plane of the other great circle[d]: wherefore the angle ſtanding on the arch, at the center of it, is right[g]; and ſo the arch itſelf is a quadrant.

[c] 4. def. 2.
[d] 5. cor. 1. 2.
[g] El. 3. def. 11.

Cor. 3. If from any point upon the ſurface of the ſphere there are drawn two arches of great circles, perpendicular to ſome other circle; that point is the pole of that other circle.

For the pole of it is upon each of theſe arches[f]; and therefore that very point is the pole.

[f] prop.

Cor. 4. If from any point upon the ſurface of the ſphere there be drawn an arch of a great circle, perpendicular to another great circle, and this arch be a quadrant; that point will be the pole of this other great circle.

For the pole of this circle muſt be upon that arch[f]: if therefore the point ſuppoſed is not the pole, an arch, either greater or leſſer than that arch, muſt be a quadrant alſo[h], which is abſurd.

Cor. 5. If from any point upon the ſurface of the ſphere there be drawn two arches of great circles, and theſe arches be quadrants; that point is the pole of the great circle paſſing through the ends of the quadrants.

For the angles at the center of the ſphere, ſtanding on thoſe quadrants, are right: wherefore each of the quadrants will be perpendicular to the great circle paſſing through their ends[k]; and ſo the point ſuppoſed is the pole of this great circle[l].

Figure II.

Cor. 6. If from any point upon the ſurface of the ſphere there be drawn two arches of great circles, E C, E D, to the circumference of another great circle C D; and if one of thoſe arches, E C, be perpendicular to the circle C D, and the other E D be a quadrant: then ſome one of the ends of the quadrant, E or D, will be the pole of one of the great circles E C, C D, which are perpendicular to one another.

For if the arch E D be perpendicular alſo to the circle C D; it is manifeſt, that E is the pole of the circle C D[l]. But if not, let the arch B D be perpendicular to the circle C D. Becauſe then each of the arches B D, B C is perpendicular to the circle C D; B therefore is the pole of the circle C D[m], and the arch B D a quadrant[h]. But the arch D E is ſuppoſed a quadrant alſo: wherefore the point D is the pole of the circle B E C[n].

PROPOSITION III.

Theor.

Fig. II.

If without any circle of the ſphere, A B C, there be taken, upon the ſurface of the ſphere, a point D, which is not the pole of that circle; and from the point D there be drawn to the circumference of that circle, arches of great circles of the ſphere, D A, D B, D C, &c. the greateſt

of

[f] prop.

[h] 2. cor.

[k] El. 4. & 18. 11.

[l] 3. or 4. cor.

[l] 3. or 4. cor.

[m] 3. cor.

[h] 2. cor.

[n] 5. cor.

of theſe arches is the arch D A, which paſſes through the pole of the circle A B C; and the arch D C, the remainder of the arch A D C, is the leaſt; and of the other arches, D B, which is nearer to the greateſt D A, is greater than D E, which is farther from it.

Let the ſtrait line A C be the common ſection of the planes of the circles A B C, A D C. From the pole F draw F G at right angles to A C; and through D draw D H parallel to F G; and join H B, H E; D A, D B, D E, D C.

Becauſe the great circle A D C paſſes through F the pole of the circle A B C, the circle A D C is perpendicular to the circle A B C[a]. And in the plane of the circle A D C there is drawn, from the pole F, the ſtrait line F G perpendicular to A C, the common ſection of theſe planes; therefore F G is perpendicular to the plane of the circle A B C[b], and will paſs through its center[c]: and ſo G is the center of the circle A B C. And becauſe F G is perpendicular to the plane of the circle A B C, and D H is parallel to F G; D H alſo is perpendicular to the ſame plane[d]: wherefore each of the angles D H A, D H B, D H E, D H C is right[e]. And becauſe from the point H, which is not the center of the circle A B C, there are drawn to the circumference of it, ſtrait lines H A, H B, H E, H C; of theſe, H A, which paſſes through the center G, is the greateſt, and H C the leaſt; and of the others, H B, which is nearer to H A, is greater than H E, which is farther from it[f]. Becauſe therefore H A is greater than H B, the ſquare of H A is greater than that of H B. add in common the ſquare of H D: and the ſquares of A H, H D, (i. e. the ſquare of D A[g]), are greater than the ſquares of B H, H D, i. e. than the ſquare of B D[g]: wherefore the ſtrait line D A is greater than D B; and the angle at the center of the ſphere, ſubtended by D A, is greater than the angle at the ſame center, ſubtended by D B[h]: wherefore the arch D A alſo is greater than the arch D B[k]. In the ſame manner it may be demonſtrated, that the arch D B is greater than the arch D E, and the arch D E greater than the arch D C. And becauſe of the ſtrait lines drawn from H, H A is the greateſt, and H C the leaſt, &c. therefore of the arches alſo drawn from D, D A is the greateſt, and D C the leaſt, &c. Q. E. D.

[a] 2. 2.
[b] El. 4. def. 11.
[c] 3. cor. 3. def. 2.
[d] El. 8. 11.
[e] El. 3. def. 11.
[f] El. 7, 8, & 15. 3.
[g] El. 47. 1
[h] El. 25. 1.
[k] El. 33. 6. & A. 5.

PROPO-

PROPOSITION IV.

Theor.

Parallel circles of the ſphere are about the ſame poles; and circles of the ſphere, which are about the ſame poles, are parallel.

Fig. III.

Firſt, let the circles of the ſphere, A B C, D E F, be parallel; and let G, H be the poles of the circle A B C. Draw G H meeting the circles A B C, D E F in the points K, L; and through G H let there paſs any plane G A D H, cutting the planes of the circles in the ſtrait lines A B, D E; and alſo any other plane G C F H cutting them in the ſtrait lines C K, F L.

Becauſe G, H are the poles of the circle A B C, G H is perpendicular to the plane of the circle A B C[a], and paſſes through the center of the ſphere[b]: wherefore each of the angles G K B, G K C is right[c]. And becauſe the circles A B C, D E F are parallel, and are cut by the plane G A D H; their common ſections A B, D E are parallel[d]: therefore the angle G L E is equal to the angle G K B[e], and the angle G K B is right: wherefore the angle G L E is right alſo. For the ſame reaſon, the angle G L F is right, and the ſtrait line G H is therefore perpendicular to each of the ſtrait lines L E, L F, at the point of their ſection: and ſo it is perpendicular to the plane of the circle D E F[f]. And it has been ſhewn, that G H paſſes through the center of the ſphere: wherefore becauſe from the center of the ſphere there is drawn a ſtrait line G H perpendicular to the plane of the circle D E F, G H is the axis of that circle[g]; and G, H are the poles of it. But G, H are the poles of the circle A B C alſo: therefore the parallel circles A B C, D E F are about the ſame poles.

[a] 1. cor. 3. def. 2.
[b] 4. cor. 1. 2.
[c] El. 3. def. 11.
[d] El. 16. 11.
[e] El. 29. 1.
[f] El. 4. 11.
[g] 2. cor. 1. 2.

But let the circles of the ſphere, A B C, D E F, be about the ſame poles G, H. Draw G H; and it will be perpendicular to each of the circles A B C, D E F[a]: and therefore the circles A B C, D E F are parallel[h].

Q. E. D

[h] El. 14. 11.

Cor. 1. If through the common pole of parallel circles A B C, D E F there paſs great circles of the ſphere, G E H, G F H; the arches of the parallel circles, B C, E F, intercepted between them, are proportional to the whole circumferences A B C, D E F; and are called *ſimilar* arches.

For theſe great circles paſs through the common axis and the centers of the parallel circles[k]: ſo that the points K, L are the centers of the circles A B C, D E F. And becauſe two ſtrait lines B K, K C, meeting together, are parallel to two ſtrait lines E L, L F, meeting together[d]; and are not in the ſame plane with them: the angle B K C is equal to the angle E L F[l]. And theſe angles are at the centers of the circles A B C, D E F: therefore as the arch B C to the circumference A B C, ſo the arch E F to the circumference D E F, as may appear from El. 33. 6.

Cor. 2. The circumference of a great circle D E F is to the circumference of a leſſer circle A B C, which is parallel to it; as alſo an arch of the great circle, E F, to a ſimilar arch B C of the leſſer parallel circle; as the radius to the ſine of the diſtance of the leſſer circle from its pole G, or to the coſine of the diſtance of the leſſer circle from the great parallel circle D E F; or again, as the ſecant of the diſtance of the leſſer circle from the great circle, to the radius. And the diſtance is meaſured by an arch of a great circle perpendicular to thoſe circles.

Let a great circle G E H paſs through the common pole G and the point B: it will be perpendicular to the circles A B C, D E F[m]; and its common ſection with the circle A B C, i. e. B K, will be perpendicular to the axis G H[n]. And the arch B G will be the diſtance of the circle A B C from its pole G, and the arch B E its diſtance from the circle D E F; and B G is the complement of B E, becauſe G E is a quadrant[o]. Draw L B. And becauſe the arches E F, B C are ſimilar; as the arch E F to the arch B C, ſo is the circumference D E F to the circumference A B C. But the circumference D E F is to the circumference A B C, as the radius E L, or L B, to the radius B K, (as appears from Archim. dim. circ. 1. & El. 2. 12); and L B is to B K, as the radius to the ſine of the arch B G[p], or to the coſine of the arch B E: wherefore the circumference D E F is to the circumference A B C, and the arch E F to the arch B C, as the radius to the ſine of the arch B G, or to the coſine of the arch B E; or again, as the ſecant of the arch B E to the radius[q].

Cor. 3. The circumferences of leſſer parallel circles, as alſo their ſimilar arches, are to one another, as the coſines of their diſtances from a great circle which is parallel to each of them; or as the ſecant of the diſtance

[k] 5. cor. 1. 2.

[d] El. 16. 11.

[l] El. 10. 11.

[m] 2. 2.

[n] 5. cor. 1. 2. & El. 3. def. 11.

[o] 2. cor. 2. 2.

[p] 10. 1.

[q] 1. 1.

diſtance of the ſecond circle to the ſecant of the diſtance of the firſt circle.

This appears from the preceding cor. by reaſoning ex aequali.

Of Spherical Angles and Triangles.

PROPOSITION V.

Theor.

Figure IV.

IF from the vertex A of a ſpherical angle BAC, as from the pole, an arch of a circle, BC, be deſcribed, meeting the ſides of the angle in the points B, C; it will be as the arch BC to its circumference, ſo the ſpherical angle BAC to four right angles.

Take D the center of the circle BC; and join DA, DB, DC: and ſo AD will be the axis of the circle BC[a]. And becauſe AB, AC are great circles[b], they paſs through the center of the ſphere[c]: but they alſo paſs through A, the pole of the circle BC; wherefore they will paſs through AD, the axis of that circle[d]. AD therefore is the common ſection of the planes of the circles AB, AC; and it is perpendicular to each of the ſtrait lines DB, DC[e], which are drawn in thoſe planes: wherefore the angle BDC is the inclination of thoſe planes[f], or the ſpherical angle BAC[b]. Therefore, becauſe D is the center of the circle BC, as the arch BC to its circumference, ſo is the angle BDC, or the ſpherical angle BAC, to four right angles. Q. E. D.

Cor. If of the ſides of a ſpherical angle, poles be taken, and axes drawn; the ſpherical angle will be equal to the inclination of thoſe axes; and it will be, as the diſtance of the poles, (meaſured by an arch of a great circle), to the circumference, ſo either of theſe angles to four right angles. And the diſtance of the neareſt poles is meaſured, when the angle is leſſer than

[a] 2. cor. 3. def. 2.

[b] 7. def. 2.

[c] 4. def. 2.

[d] 5. cor. 1. 2.

[e] 2. def. 2. & El. 3. def. 11.

[f] El. 6. def. 11.

than a right angle; but that of the more remote, when the angle is greater than a right one.

Let E, F be the poles of the ſides A B, A C of a ſpherical angle B A C: through E, F draw a great circle E F B C, and let D be its center; join E D, F D, B D, C D, wherefore E D, F D will be the axes of the circles A B, A C[a]; and E F the diſtance of their poles. And becauſe the great circle B C paſſes through E the pole of the great circle A B[b]; converſely, the great circle A B paſſes through the pole of the circle B C[g], for the ſame reaſon, the circle A C alſo paſſes through the pole of the circle B C: and ſo A is the pole of the circle B C. And becauſe E is the pole of the great circle A B, the arch E B is a quadrant[h]: and for the ſame reaſon, the arch F C is a quadrant, and is equal to the arch E B. take them away from F B: and the arch E F is equal to B C: and therefore the angle E D F, (i. e. the inclination of the axes E D, D F), is equal to the angle B D C[k], i. e. to the ſpherical angle B A C, as has been ſhewn in the prop. Therefore, becauſe as the arch B C to the circumference, ſo is the ſpherical angle B A C to four right angles[l]; the ſame ratio will the arch E F alſo, (i. e. the diſtance of the poles E, F), have to the circumference[m]; and the angle E D F, (i. e. the inclination of the axes E D, D F), to four right angles[m].

[a] 2. cor. 3. def 2.

[b] 7. def. 2.

[g] 1. cor. 2. 2.

[h] 2. cor. 2. 2.

[k] El. 27. 3.

[l] prop.

[m] El. 7. 5.

Theor.

PROPOSITION VI.

An arch of a great circle, ſtanding upon an arch of another great circle, makes the adjacent angles equal to two right angles; and if thoſe angles be equal, each of them is a right one.

Figure V.

Let A B, C D be arches of great circles; and from the pole B let there be deſcribed an arch of a great circle, C A D, meeting the arch C D in the points C, D.

Becauſe, as the arch C A to the circumference, ſo is the angle C B A to four right angles[a]; and as the arch A D to the circumference, ſo is the angle A B D to four right angles[a]: it will be, as the arch C A D to the circumference, ſo the angles C B A, A B D together, to four right angles[b].

[a] 5. 2.

[b] El. 24. 5.

But the arch C A D is half the circumference[c]; therefore the angles C B A, A B D together are equal to two right angles[d]. And if the angles C B A, A B D are equal; then, becauſe it has been demonſtrated, that, taken together, they are equal to two right angles, it is manifeſt, that each of them is right. Q. E. D.

[c] 1. cor. 4. def. 2.
[d] El. D. 5.

Cor. 1. If two arches of great circles cut one another, the vertical angles are equal; and all the four angles, taken together, are equal to four right angles.

Cor. 2. In a ſpherical triangle A B C, every angle, as B A C, is leſſer than two right angles.

For each of the ſides C A, C B is leſſer than a ſemicircumference[e]: being therefore produced, they will meet again in a point D, which will be without the ſides C A, C B themſelves[e]. And it is without the ſide A B alſo, becauſe each of the arches A B, A D is leſſer than a ſemicircumference[e]: wherefore the point D is without the triangle A B C, and the angle B A D without the angle B A C, and adjacent to it. But theſe two are together equal to two right angles[f]: wherefore the angle B A C is leſſer than two right angles.

[e] 9. def. 2.
[f] prop.

Cor. 3. In a ſpherical trilateral figure, having one of its ſides greater than a ſemicircumference, the angle, oppoſite to that ſide, is greater than two right angles; as it may be demonſtrated in the ſame manner.

PROPOSITION VII.

Theor.

Figure VI.

If of the ſides of a ſpherical triangle A B C, the poles D, E, F be taken, ſuch as are the neareſt to the angles oppoſite to thoſe ſides; and through theſe poles, arches of great circles, D E, E F, F D, leſſer than ſemicircumferences, be drawn: theſe arches will conſtitute another ſpherical triangle D E F, called *ſupplemental* of the former. The ſides of this triangle, E F, F D, D E, are to the circumference, as the ſupplements of the oppoſite angles A, B, C, in the triangle A B C, to four right angles; and its angles D, E, F are to four right angles, as the ſupplements of the oppoſite ſides BC, CA, AB, in the triangle ABC, to the circumference.

Produce AB, BC, CA; or, if neceſſary, produce DE, EF, FD; and let AB meet EF, FD in G, H; and BC meet ED, DF in K, L; and CA meet DE, EF in M, N.

Becauſe EF paſſes through F the pole of the arch AG; converſely, AG paſſes through the pole of the arch EF[a]. For the ſame reaſon, AN paſſes through the pole of the arch EF: wherefore A is the pole of the arch EF. And the ſame way it may be demonſtrated, that B is the pole of FD, and C the pole of DE.

And becauſe E is the pole of the arch AN, the arch EN is a quadrant[b]: and for the ſame reaſon, the arch FG is a quadrant, therefore the arches EN, FG, or GN, EF, are equal to a ſemicircumference: and ſo EF is the ſupplement of GN. And becauſe, as the arch GN to the circumference, ſo is the angle BAC to four right angles[c], (for A is the pole of GN); and as a ſemicircumference to the whole circumference, ſo are two right angles to four[d]: as therefore the ſupplement of the arch GN, i. e. as EF, to the circumference, ſo is the ſupplement of the oppoſite angle BAC, in the triangle ABC, to four right angles[e]. The ſame way it may be demonſtrated, that as the ſides FD, DE to the circumference; ſo are the ſupplements of the oppoſite angles B, C, in the triangle ABC, to four right angles.

Alſo, becauſe B is the pole of the arch DL, the arch BL is a quadrant[b]: and for the ſame reaſon, CK is a quadrant, therefore the arches BL, CK, or BC, KL, are equal to a ſemicircumference; and ſo KL is the ſupplement of BC. Wherefore, becauſe D is the pole of KL, as the angle EDF to four right angles, ſo is the arch KL, i. e. the ſupplement of the oppoſite ſide BC, in the triangle ABC, to the circumference[c]. And the ſame way it may be demonſtrated, that as the angles E, F to four right ones, ſo are the ſupplements of the oppoſite ſides CA, AB, in the triangle ABC, to the circumference. Q. E. D.

Cor. Converſely, the triangle ABC is ſupplemental of the triangle DEF.

[a] 1. cor. 2. 2.

[b] 2. cor. 2. 2.

[c] 5. 2.

[d] El. C. 5.

[e] El. 1. cor. 24. 5.

[b] 2. cor. 2. 2.

[c] 5. 2.

PROPO-

PROPOSITION VIII.

Theor.

Any two ſides of a ſpherical triangle are greater than the third.

Let A B C be a ſpherical triangle. From D, the center of the ſphere, draw D A, D B, D C.

Fig. VII.

Becauſe the angles B D A, A D C, C D B are at the centers of great circles [a], and therefore of equal ones [b]; as the angle B D A to the angle B D C, ſo is the arch A B to B C [c]; and as the angle A D C to the angle B D C, ſo is the arch A C to B C [c]: wherefore as the angles B D A, A D C together to the angle B D C, ſo are B A, A C together to B C [d]. But the angles B D A, A D C together are greater than the angle B D C [e]; becauſe theſe three contain a ſolid angle at D, the center of the ſphere: therefore alſo the ſides B A, A C together are greater than the ſide B C [f].

Q. E. D.

Cor. The difference of any two ſides of a ſpherical triangle is leſſer than the third ſide.

For, if not, add the leſſer ſide in common: then the leſſer ſide, together with the difference, i. e. the greater ſide, will not be leſs than the other two ſides of the triangle; which is impoſſible [g].

[a] 9. def. 2.
[b] 4. def. 2.
[c] El. 33. 6.
[d] El. 24. 5.
[e] El. 20. 11.
[f] El. A. 5.
[g] prop.

PROPOSITION IX.

Theor.

The three ſides of a ſpherical triangle are together leſſer than the circumference of a great circle.

Let A B C be a ſpherical triangle. From D, the center of the ſphere, draw D A, D B, D C.

Fig. VII.

Becauſe the angles B D A, A D C, C D B are at the centers of great [a], and equal circles [b]; as the angle B D A to four right angles, ſo is the arch A B to the circumference; and as the angle A D C to four right angles,

[a] 9. def. 2.
[b] 4. def. 2.

ſo is the arch A C to the circumference; and alſo as the angle C D B to four right angles, ſo is the arch C B to the circumference: as therefore the angles B D A, A D C, C D B together, to four right angles, ſo are B A, A C, C B together, to the circumference [c]. But thoſe angles are together leſſer than four right angles [d]; becauſe they contain a ſolid angle at D, the center of the ſphere: therefore alſo the three ſides B A, A C, C B are together leſſer than the circumference of a great circle [e]. Q. E. D.

[c] El. 2. cor. 24. 5.

[d] El. 21. 11.

[e] El. A. 5.

Theor.

PROPOSITION X.

In a ſpherical triangle, equal ſides ſubtend equal angles; and equal angles are ſubtended by equal ſides.

Fig. VII.

Let A B C be a ſpherical triangle; and firſt, let its ſides A C, A B be equal. I ſay, that the angle A B C is equal to the angle A C B.

Biſect the baſe B C in E; and through A, E draw an arch of a great circle, A E [a]; and from D, the center of the ſphere, draw D A, D B, D E, D C.

Becauſe the arch A B is equal to the arch A C, the angle B D A is equal to the angle A D C [b]: for the ſame reaſon, the angle B D E is equal to the angle E D C. And the angle A D E is common: wherefore there are two ſolid angles at D, the center of the ſphere; and each of them is contained by three plane angles equal to one another, each to each: therefore the planes, in which the equal angles are, have the ſame inclination to one another [c]. Wherefore the inclination of the planes A D B, B D E is the ſame with the inclination of the planes A D C, C D E. But the inclinations of theſe planes are the ſpherical angles A B C, B C A [d]: therefore alſo the angle A B C is equal to the angle A C B.

[a] 1. cor. 1. 2.

[b] El. 27. 3.

[c] El. A. 11.

[d] 7. def. 2.

Fig. VI.

But let the angle A B C be equal to the angle A C B. I ſay, the ſide A C is equal to the ſide A B.

Take the triangle D E F, ſupplemental of the triangle A B C.

And becauſe the angles A B C, B C A are equal, their ſupplements alſo are equal. But the ſupplements of theſe angles are to one another, as the ſides F D, D E, which are oppoſite to them [e]: therefore the ſides F D,

[e] 7. 2. & El. 22. 5.

DE are equal[f]; and, by the preceding demonſtration, the angles E, F, ſubtended by them, are equal. But as the angles E, F to one another, ſo are the ſupplements of the ſides AC, AB in the triangle ABC[e]: therefore the ſupplements of the ſides AC, AB are equal[f]; and ſo AC, AB themſelves are equal. Q. E. D.

[f] El. A. 5.
[e] 7. 2. & El. 22. 5.

PROPOSITION XI. Theor.

In a ſpherical triangle, the greater angle is ſubtended by the greater ſide; and the greater ſide ſubtends the greater angle.

Let ABC be a ſpherical triangle, and its angle BAC greater than B. I ſay, the ſide BC is greater than AC. *Fig.* VIII.

Make the angle BAD equal to the angle B: therefore AD will fall within the triangle ABC, ſince the angle BAC is greater than BAD; and ADB, ADC will be ſpherical triangles[a]. And becauſe the angle B is equal to the angle BAD, AD is equal to BD[b]; and AD, DC together are equal to BC. But AD, DC are together greater than AC[c]: wherefore BC alſo is greater than AC.

[a] 3. & 2. cor. 6. 2.
[b] 10. 2.
[c] 8. 2.

And if the ſide BC be greater than AC, the angle BAC will alſo be greater than B; which may be demonſtrated as El. 19. 1.

PROPOSITION XII. Theor.

In a ſpherical triangle, if two ſides are together equal to a ſemicircumference; the two angles, ſubtended by them, are together equal to two right angles: and if two ſides are together greater, or leſſer, than a ſemicircumference; the two angles, ſubtended by them, are together greater, or leſſer, than two right angles: And converſely.

Figure V.

Let A B C be a ſpherical triangle; and B A, A C its two ſides. Produce the arches C A, C B[a], till they meet again in D[b].

And firſt, let B A, A C be together equal to a ſemicircumference, wherefore they are equal to the arch C A D[b]. take away the common C A: then A B is equal to A D; and ſo the angle D is equal to the angle A B D[c]. But the angle D is equal to the angle C; becauſe they are the inclinations of the ſame planes, viz. of the planes of the circles C A D, C B D[d]: wherefore alſo the angle C is equal to the angle A B D. Add in common the angle A B C: and the angles C and A B C will be equal to the angles A B D, A B C. But the angles, A B D, A B C are together equal to two right angles[e]: wherefore the angles C and A B C alſo are together equal to two right angles.

But let the ſides B A, A C be together greater than a ſemicircumference, i. e. greater than the arch C A D[b]. Then A B is greater than A D; and the angle D, or C[d], greater than the angle A B D[f]; and the two angles C and A B C together, greater than the two A B D, A B C together, i. e. greater than two right angles[e]. In the ſame manner it may be demonſtrated, that if the ſides B A, A C are leſſer than a ſemicircumference, the angles C and A B C alſo will be together leſſer than two right angles.

And if, converſely, the angles C and A B C be together greater, or leſſer, than two right angles, or equal to them; the ſides B A, A C will alſo be together greater, or leſſer, than a ſemicircumference, or equal to it: which may be demonſtrated, if the ſteps of the preceding demonſtration be taken in an inverſe order. Q. E. D.

Cor. 1. Half the ſum of two angles of a ſpherical triangle is of the ſame affection with half the ſum of the two ſides, ſubtending them: and converſely. And angles, as well as arches, are ſaid to be *of the ſame affection* with angles, or arches, when they are either of 90°, or both leſſer than 90°, or both greater than 90°, but leſſer than 180°: otherwiſe they are ſaid to be *of different affection*.

Cor. 2. If two ſides of a ſpherical triangle are equal; each of the angles, ſubtended by them, is of the ſame affection with either of them: and if two angles are equal, each of the ſides, ſubtending them, is of the ſame affection with either of them[g].

[a] 9. def. 2.

[b] 1 cor. 4. def. 2.

[c] 10. 2.

[d] 7. def. 2.

[e] 6. 2.

[b] 1. cor. 4. def. 2.

[d] 7. def. 2.

[f] 11. 2.

[g] 10. 2. & 1. cor. prop.

PROPOSITION XIII.

Theor.

The three angles of a ſpherical triangle are greater than two right angles, and leſſer than ſix: and the difference between the ſum of any two of them and the third angle, is leſſer than two right angles.

Figure VI.

Let A B C be a ſpherical triangle. Take the triangle D E F ſupplemental of it.

Firſt, becauſe the three ſides E F, F D, D E of the triangle D E F are leſſer than the circumference of a great circle[a]; and theſe ſides are to the circumference, as the ſupplements of the angles A, B, C to four right angles[b]: therefore the ſupplements of the angles A, B, C alſo are leſſer than four right angles[c]. But the ſupplements of the angles A, B, C, together with thoſe angles themſelves, are equal to ſix right angles[d]: wherefore the angles A, B, C alone are greater than two right angles. And it is evident, that the angles A, B, C are leſſer than ſix right angles[d].

The ſecond part is manifeſt, when the ſum of two angles is not greater than the third angle. But let the ſum of the angles A, B be greater than the angle C. Becauſe the two ſides E F, F D of the triangle D E F are greater than the third D E[e]; and theſe ſides are to one another, as the ſupplements of the angles A, B, C[f]: therefore the ſupplements of the angles A, B are greater than the ſupplement of C[c]. Wherefore the exceſs of four right angles above the ſupplements of the angles A, B, is leſſer than the exceſs of the ſame four right angles above the ſupplement of C. But the exceſs of four right angles above the ſupplements of the angles A, B, is equal to thoſe angles themſelves; and the exceſs of four right angles above the ſupplement of the angle C, is equal to this ſame angle C together with two right angles: therefore the angles A, B are leſſer than two right angles together with the angle C; whence the exceſs of the angles A, B above C is leſſer than two right angles. Q. E. D.

Cor. 1. Any two angles of a ſpherical triangle are together greater than the exterior oppoſite angle.

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[a] 9. 2.

[b] 7. 2. & El. 2. cor. 24. 5.

[c] El. A. 5.

[d] 2. cor. 6. 2.

[e] 8. 2.

[f] 7. 2. & El. 24. & 22. 5.

Book II.

Figure V.

For the three angles of the triangle A B C are together greater than two right angles[g]; and ſo they are greater than the angles A B D, A B C together[h]. take away the angle A B C: and the two remaining angles A, C are together greater than the angle A B D.

[g] prop.
[h] 6. 2.

Cor. 2. The ſum of two angles of a ſpherical triangle is greater than the ſupplement of the third angle, but leſſer than two right angles together with the third angle; and their difference is leſſer than the ſupplement of the third angle.

For firſt, the ſupplement of the third angle is the exterior oppoſite angle; and the other two angles are greater than it[k].

[k] 1. cor.

And the ſecond has been ſhewn in the prop.

Thirdly, becauſe the difference between the ſum of the two angles A, C, and the third angle B, i. e. the difference of the angles A, B, together with the third C, is leſſer than two right angles[g]: therefore the difference of the angles A, B is leſſer than the exceſs of two right angles above the angle C, i. e. it is leſſer than the ſupplement of C.

Cor. 3. In a right angled ſpherical triangle, the ſum of the two angles, excluding the right one, is greater than one right angle, and leſſer than three right angles; and their difference is leſſer than one right angle[l].

[l] 2. cor.

Theor.

PROPOSITION XIV.

In a rightangled ſpherical triangle, each of the two angles, excluding the right one, is of the ſame affection with the ſide ſubtending it; and each of the two ſides about the right angle is of the ſame affection with the angle ſubtended by it.

Figure IX.
X.

Let A B C be a rightangled ſpherical triangle, and its angle A right. I ſay, the angle A C B is of the ſame affection with the ſide A B.

Take A D a quadrant; and through C, D draw an arch of a great circle, C D[a].

[a] 1. cor. 1. 2.

And firſt, let the ſide A B be leſſer than a quadrant. Becauſe the angle D A C is right, D A is perpendicular to A C, and it is alſo a quadrant: therefore the point D is the pole of the arch A C[b]: wherefore D C is a qua-

[b] 4. cor. 2. 2.

a quadrant [c], and the angle D C A a right one [d]. And becauſe A B is leſſer than a quadrant, and A D is a quadrant; A B is leſſer than A D: wherefore the point D is without the triangle A B C; and the whole C D, being leſſer than a ſemicircumference, is without the triangle A B C. Whence the angle A C B is leſſer than the angle A C D, i. e. leſſer than a right angle. In like manner it is demonſtrated, that if the ſide A B be greater than a quadrant, the angle A C B is greater than a right one. And if A B be a quadrant, the point B is the pole of the arch A C [b]; and therefore the angle A C B is a right one [d].

[c] 2. cor. 2. 2.
[d] 2. 2.
Figure X.
[b] 4. cor. 2. 2.

I ſay alſo, that the ſide A B is of the ſame affection with the angle A C B, ſubtended by it.

Figure IX.
X.

Take A C D a right angle. D therefore is the pole of the arch A C [e]; and each of the arches A D, D C a quadrant [c]. If then the angle A C B is leſſer than a right one, C D will fall without the triangle A B C; and A B will be leſſer than A D, i. e. leſſer than a quadrant. If the angle A C B is greater than a right one, C D will be within the triangle A B C; and A B will be greater than A D, or greater than a quadrant. And if the angle A C B is a right one, B will be the pole of A C [e], and A B a quadrant [c]. Q. E. D.

[e] 3. cor. 2. 2.
[e] 3. cor. 2. 2.
[c] 2. cor. 2. 2.

Cor. In a quadrantal ſpherical triangle, each of the two ſides, excluding the quadrant, is of the ſame affection with the angle, ſubtended by it; and each of the two angles adjacent to the quadrant, is of the ſame affection with the ſide, ſubtending it [f].

[f] prop. & 7. 2.

PROPOSITION XV.

Theor.

In a rightangled ſpherical triangle, if the two ſides about the right angle be of the ſame affection, the hypotenuſe is leſſer than a quadrant; but if thoſe ſides be of different affection, the hypotenuſe is greater than a quadrant; and if one of them be a quadrant, the hypotenuſe is a quadrant alſo.

Let

Figure IX. X.

Let A B C be a rightangled ſpherical triangle, and its angle A right. Produce the ſides A B, A C, till they meet again in E; and take the quadrants A D, A F; and through C, D draw an arch of a great circle, C D.

Figure IX.

And firſt, let the ſides A B, A C be of the ſame affection, and each of them leſſer than a quadrant.

[a] 4. cor. 2. 2.
[b] 2. cor. 2. 2.

Becauſe A F is a quadrant, and at right angles to A B; F is the pole of A B[a], for the ſame reaſon, D is the pole of A C: wherefore D C is a quadrant[b]. And becauſe A B is leſſer than a quadrant, i. e. leſſer than A D; C D will fall without the triangle A B C. Therefore, A C being leſſer than the quadrant A F, from the point C, which is not the pole of the arch A D E, there are drawn arches of great circles, C E, C D, C B; and the arch C D is nearer to the arch C E, which paſſes through F, the pole of the arch A D E, than the arch C B: C B then is leſſer than C D[c], i. e. leſſer than a quadrant. In like manner it is demonſtrated, that in the rightangled ſpherical triangle B C E, whoſe ſides E B, E C are greater than quadrants, the hypotenuſe C B is leſſer than a quadrant.

[c] 3. 2.

Figure X.

But let the ſides A B, A C be of different affection; and A B greater than a quadrant, and A C leſſer. The points F, D are, as before, the poles of the arches A D E, A C; and D C is a quadrant. And becauſe A B is greater than a quadrant, i. e. greater than A D; C B is nearer to the arch C E, which paſſes through F the pole of the arch A D E, than C D: and ſo C B is greater than C D[c], i. e. greater than a quadrant.

And if one of the ſides, as A D, of the rightangled ſpherical triangle A C D be a quadrant, the point D will be the pole of the arch A C[a]; and ſo the hypotenuſe D C is a quadrant[b]. Q. E. D.

[a] 4. cor. 2. 2.
[b] 2. cor. 2. 2.

Cor. 1. In a rightangled ſpherical triangle, if the two angles, excluding the right one, be of the ſame affection, the hypotenuſe is leſſer than a quadrant; but if thoſe angles be of different affection, the hypotenuſe is greater than a quadrant; and if one of them be a right angle, the hypotenuſe is a quadrant[d].

[d] prop. &c. 14. 2.

Cor. 2. Converſely, in a rightangled ſpherical triangle, if the hypotenuſe be leſſer than a quadrant, the two ſides about the right angle, and therefore the angles ſubtended by them[e], are of the ſame affection; and if it be greater than a quadrant, thoſe ſides, and angles[e], are of different affection;

[e] 14. 2.

ſection; and if a quadrant, one of thoſe ſides is a quadrant alſo[f], and the angles ſubtended by it, a right one[e].

[f] 6. & 2. cor. 2. 2.

[e] 14. 2.

Cor. 3. If the hypotenuſe be leſſer, or greater, than a quadrant; each of the two ſides about the right angle, is of the ſame, or of different, affection with the angle adjacent to it: and converſely.

Cor. 4. In a quadrantal ſpherical triangle, if the two angles, adjacent to the quadrant, or the two ſides ſubtending them[g], be of the ſame affection; the angle ſubtended by the quadrant, is greater than a right one: and if thoſe two angles, or thoſe two ſides, be of different affection; the angle ſubtended by the quadrant, is leſſer than a right one: and converſely[h].

[g] cor. 14. 2.

[h] 7. 2.

PROPOSITION XVI.

Theor.

Figure XI. XII.

In a ſpherical triangle ABC, whoſe two angles at the baſe, B, C, are oblique, if AD, the perpendicular from the vertex A upon the baſe BC, fall within the triangle; the angles B, C are of the ſame affection: but if the perpendicular AD fall without the triangle, the angles B, C are of different affection.

Figure XI.

For if AD falls within the triangle, becauſe ADB, ADC are right-angled ſpherical triangles; each of the angles B, C is of the ſame affection with AD[a]: wherefore they are themſelves of the ſame affection. And if AD falls without the triangle, each of the angles B and ACD is of the ſame affection with AD[a], and ſo themſelves are of the ſame affection. But the angles ACD and ACB are of different affection[b]: wherefore alſo the angles B and ACB are of different affection.

[a] 14. 2.

Fig. XII.

[b] 6. 2.

Cor. Converſely, if the angles B, C be of the ſame affection, the perpendicular falls within the triangle: for if it ſhould fall without, thoſe angles would be of different affection[c]. And if the angles B, C be of different affection, the perpendicular falls without the triangle: for if it ſhould fall within, thoſe angles would be of the ſame affection[c].

[c] prop.

PROPOSI-

Theor.

PROPOSITION XVII.

Figure XI. XII.

In a ſpherical triangle ABC, wherein the two angles at the baſe, B, C, are oblique, and from the vertex A upon the baſe BC the perpendicular AD is drawn; if the ſides BA, AC be of the ſame affection; the ſegments of the baſe alſo, DB, DC, and the angles at the vertex, BAD, DAC, are of the ſame affection: but if the ſides BA, AC be of different affection; the ſegments of the baſe alſo, and the angles at the vertex, are of different affection: And converſely.

For, if poſſible, let the ſides BA, AC be of the ſame affection; but the ſegments DB, DC be of different affection.

And firſt, let BA, AC, DB be each of them leſſer than a quadrant; but DC greater than a quadrant. And becauſe in the rightangled ſpherical triangle ABD, the hypotenuſe AB is leſſer than a quadrant; the ſides BD, DA are of the ſame affection [a]. And BD is leſſer than a quadrant; wherefore alſo AD is leſſer than a quadrant. But becauſe DC is, by the ſuppoſition, greater than a quadrant; therefore in the rightangled ſpherical triangle ACD, the ſides AD, DC are of different affection: wherefore the hypotenuſe AC is greater than a quadrant [b]. But it is alſo leſſer, by the ſuppoſition: which is impoſſible. DC therefore is not greater than a quadrant, but leſſer; and ſo it is of the ſame affection with DB. And becauſe the ſegments DB, DC are of the ſame affection; the angles at the vertex alſo, DAB, DAC, are of the ſame affection [c].

In like manner the prop. is demonſtrated in every one of its caſes.

[a] 2. cor. 15. 2.

[b] 15. 2.

[c] 14. 2.

PROPOSITION XVIII. Theor.

IN a rightangled ſpherical triangle, as the ſine of the hypotenuſe to the radius, ſo is the ſine of either of the two ſides about the right angle, to the ſine of the angle oppoſite to that ſide.

In rightangled ſpherical triangles, ſuch as have more than one right angle, this is manifeſt: for the ſides oppoſite to the right angles, are quadrants[a]; and the remaining ſide is to the circumference, as the angle oppoſite to it, to four right angles[b]: wherefore each ſine of one of the quadrants is equal to the radius[c], and the ſine of the other quadrant is equal to the ſine of the right angle oppoſite to it[c]; and alſo the ſine of the remaining ſide is equal to the ſine of the angle which is oppoſite to it[d].

But let the triangle A B C have one right angle only; and let it be the angle A. Wherefore none of its three ſides is a quadrant[e]. I ſay, that as the ſine of the hypotenuſe B C to the radius, ſo is the ſine of the ſide A C to the ſine of the angle B.

Fig. XIII. XIV.

Firſt, let A C be leſſer than a quadrant. And let D be the center of the ſphere: join D A, D B, D C; and from C draw the ſtrait line C E perpendicular to D B, which therefore is the ſine of the hypotenuſe B C. From E draw, in the plane A D B, the ſtrait line E F at right angles to D B. E F will meet A D, ſince the arch A B is not a quadrant. Let them meet in F: join F C.

Becauſe each of the ſtrait lines C E, E F, in the planes C D B, D B A, is at right angles to D E, the common ſection of thoſe planes; the angle C E F therefore is the inclination of thoſe planes[f]; or the ſpherical angle A B C[g]. Again, becauſe the ſtrait line D E is at right angles to each of the ſtrait lines CE, EF; it is therefore at right angles to the plane CEF[h]: wherefore alſo the plane ADB, paſſing through DE, is at right angles to the plane CEF[k]; and converſely, the plane CEF is at right angles to the plane ADB. But becauſe the ſpherical angle BAC is right, the plane ADC alſo is at right angles to the ſame plane ADB, as may appear from 8. def. and the Elements: therefore alſo the common ſection of the planes CEF, ADC, i. e. the ſtrait line CF, is at right angles to the plane ADB[l]: wherefore each of the angles CFA, CFE is right[m]; and CF is the ſine of the ſide CA, and the triangle CFE rightangled. Becauſe then in the rightangled plane triangle CFE, as the hypotenuſe CE to the radius, ſo is the ſide CF to the ſine of the angle CEF[n]: it will be in the rightangled ſpherical triangle ABC alſo, as CE, the ſine of the hypotenuſe BC, to the radius, ſo CF, the ſine of the ſide AC, to the ſine of the angle CEF, i. e. to the ſine of the ſpherical angle B.

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[a] 14. 2.

[b] 3. cor. 2. 2. & 5. 2.

[c] 2. cor. 4. def. 1.

[d] 4 cor. 9. def. 1.

[e] 14. & 15. 2.

[f] El. 6. def. 11.

[g] 7. def. 2.

Figure IX. X.

And if in the rightangled ſpherical triangle BCE, the ſide CE be greater than a quadrant; produce EC, EB, till they meet again in A. Therefore in the rightangled ſpherical triangle ABC, as the ſine of the hypotenuſe BC to the radius, ſo is the ſine of the ſide AC to the ſine of the angle ABC, as has been demonſtrated: wherefore in the rightangled ſpherical triangle BCE alſo, as the ſine of the hypotenuſe BC to the radius, ſo is the ſine of the ſide CE[o] to the ſine of the angle CBE[o]. Q. E. D.

Cor. In a quadrantal ſpherical triangle, as the ſine of the angle ſubtended by the quadrant, to the radius; ſo is the ſine of either of the two angles adjacent to the quadrant, to the ſine of the ſide oppoſite to that angle[p].

Theor.

PROPOSITION XIX.

In a rightangled ſpherical triangle, as the ſine of either of the two ſides about the right angle, to the radius, ſo is the tangent of the other ſide to the tangent of the angle oppoſite to that ſide.

[h] El. 4. 11.

[k] El. 18. 11.

[l] El. 19. 11.

[m] El. 3. def. 11.

[n] 10. 1.

[o] 6. cor. 9. def. 1.

[p] prop. & 7. 2.

In rightangled ſpherical triangles, ſuch as have more than one right angle, this is manifeſt: for the ſides oppoſite to the right angles, are quadrants [a]; and the remaining ſide is to the circumference, as the angle oppoſite to it, to four right angles [b]: wherefore each ſine of either of the quadrants is equal to the radius [c], and the tangent of the remaining ſide is equal to the tangent of the angle oppoſite to that ſide [d].

Fig. XIII. XIV.

But let the triangle A B C have one right angle only; and let it be the angle A. Wherefore none of its three ſides is a quadrant [e]. I ſay, that as the ſine of the ſide A B to the radius, ſo is the tangent of the other ſide A C to the tangent of the angle B.

Firſt, let A C be leſſer, than a quadrant. And let D be the center of the ſphere: join D A, D B, D C; and from A draw the ſtrait line A G perpendicular to D B, which therefore is the ſine of the ſide A B. From G draw, in the plane B D C, the ſtrait line G H at right angles to D B. G H will meet D C, ſince the arch B C is not a quadrant. Let them meet in H: join H A.

Therefore the angle A G H is the inclination of the planes A D B, B D C; or the ſpherical angle A B C [f]; and the ſtrait line H A is perpendicular to the plane A D B; as it may be demonſtrated the ſame way, as was done in the preceding prop. concerning the angle C E F and the ſtrait line C F. Wherefore each of the angles H A D, H A G is right [g]; and A H is the tangent of the ſide A C, and the triangle H A G rightangled. Becauſe then in the rightangled plane triangle H A G, as the ſide G A to the radius, ſo is the other ſide A H to the tangent of the angle A G H [h]: it will be in the rightangled ſpherical triangle A B C alſo, as G A, the ſine of the ſide A B, to the radius, ſo A H, the tangent of the other ſide A C, to the tangent of the angle A G H, i. e. to the tangent of the ſpherical angle B.

Figure IX. X.

And if in the rightangled ſpherical triangle B C E, the ſide C E be greater than a quadrant; produce E C, E B, till they meet again in A. Therefore in the rightangled ſpherical triangle A B C, as the ſine of the ſide A B to the radius, ſo is the tangent of the other ſide A C to the tangent of the angle A B C, as has been demonſtrated: wherefore in the rightangled ſpherical triangle B C E alſo, as the ſine of the ſide E B [k] to the radius, ſo is the tangent of the other ſide C E [k] to the tangent of the angle C B E [k].

Q. E. D.

[a] 14. 2.

[b] 3. cor. 2. 2. & 5. 2.

[c] 2. cor. 4. def. 1.

[d] 4. cor. 9. def. 1.

[e] 14. & 15. 2.

[f] 7. def. 2.

[g] El. 3. def. 1.

[h] 10. 1.

[k] 6. cor. 9. def. 1.

Cor. 1. In a quadrantal ſpherical triangle, as the ſine of either of the two angles adjacent to the quadrant, to the radius; ſo is the tangent of the other of thoſe angles to the tangent of the ſide oppoſite to it[l].

[l] prop. & 7. 2.

Cor. 2. In a rightangled ſpherical triangle ABC, either of the two ſides about the right angle, as AC, if leſſer than a quadrant, has a leſſer ratio to the circumference, than the angle oppoſite to it, ABC, has to four right angles: but if the ſide AC be greater than a quadrant, it has a greater ratio to the circumference, than the angle ABC to four right angles.

For Sin. AB : Rad. :: tang. AC : tang. ABC[m]; whence the ſine of AB being leſſer than the radius, the cor. is manifeſt from cor. 4. 1.

[m] prop.

Cor. 3. In a quadrantal ſpherical triangle, either of the ſides, excluding the quadrant, if leſſer than a quadrant, has a greater ratio to the circumference, than the angle oppoſite to it has to four right angles: but if the ſide be greater than a quadrant, it has a leſſer ratio to the circumference, than the angle oppoſite to it has to four right angles[n].

[n] 2. cor. & 7. 2.

Theor.

PROPOSITION XX.

In a rightangled ſpherical triangle, as the coſine of either of the two ſides about the right angle, to the radius, ſo is the coſine of the hypotenuſe to the coſine of the remaining ſide.

Fig. XV.

Conſtruction for this prop. and the three following.

Let ABC be a rightangled ſpherical triangle, and its angle A right. From one of the points B, C, e. g. from B, as the pole, deſcribe an arch of a great circle, DEF, meeting the ſides AB, BC, CA in the points D, E, F.

Becauſe B is the pole of the arch DF; each of the arches BD, BE is a quadrant[a]; and each perpendicular to the arch DF[b]. Therefore the angle CEF is right, and the triangle CEF rightangled; and AD is the complement of the ſide AB, and CE the complement of the hypotenuſe BC: alſo the arch ED is to the circumference, as the angle ABC

[a] 2. cor. 2. 2.
[b] 2. 2.

to four right angles [c]. And becauſe the arch B D is perpendicular to D F; converſely, the arch D F is perpendicular to B D. But alſo the arch A F is perpendicular to the ſame B D, becauſe the angle B A C is right: therefore from the point F there are drawn two arches F A, F D, each of which is perpendicular to the arch B D: and ſo F is the pole of the arch B D [d]; and therefore each of the arches F A, F D is a quadrant [a]; and F C is the complement of the ſide C A, and F E the complement of the arch E D: alſo, the angle A F D, or C F E [e], is to four right angles, as the arch A D, i. e. the complement of the ſide A B, to the circumference [c].

[c] 5. 2.

[d] 3. cor. 2. 2.

[a] 2. cor. 2. 2.

[e] 1. cor. 6. 2.

Then in the rightangled ſpherical triangle C E F, as the ſine of the hypotenuſe C F to the radius, ſo is the ſine of the ſide C E to the ſine of the angle C F E [f]: i. e. in the rightangled ſpherical triangle A B C, as the coſine of one of its ſides, A C, to the radius, ſo is the coſine of the hypotenuſe B C to the coſine of the other ſide A B [g].

[f] 18. 2.

[g] 4. cor. 9. def. 1.

Cor. In a quadrantal ſpherical triangle, as the coſine of either of the two angles adjacent to the quadrant, to the radius; ſo is the coſine of the angle ſubtended by the quadrant, to the coſine of the remaining angle [h].

[h] prop. & 7. 2.

PROPOSITION XXI.

Theor.

In a rightangled ſpherical triangle, as the coſine of either of the two angles, excluding the right, to the radius, ſo is the cotangent of the hypotenuſe to the cotangent of the ſide adjacent to that angle.

Fig. XV.

For in the rightangled ſpherical triangle C E F, as the ſine of the ſide F E to the radius, ſo is the tangent of the other ſide C E, to the tangent of the angle C F E [a]: i. e. in the rightangled ſpherical triangle A B C, as the coſine of the angle A B C [b] to the radius, ſo is the cotangent of the hypotenuſe B C to the cotangent of the ſide A B [b], adjacent to the angle A B C. The ſame will be demonſtrated of the angle A C B and the ſide A C, if from C, as the pole, an arch of a great circle be deſcribed, on the ſide of A B.

[a] 19. 2.

[b] 4. cor. 9. def. 1.

Cor. In a quadrantal ſpherical triangle, as the coſine of either of the two ſides, excluding the quadrant, to the radius; ſo is the cotangent of the angle ſubtended by the quadrant, to the cotangent of the angle which is adjacent to that ſide[c].

[c] prop. & 7. 2.

Theor.

PROPOSITION XXII.

In a rightangled ſpherical triangle, as the coſine of either of the two ſides about the right angle, to the radius, ſo is the coſine of the angle oppoſite to that ſide, to the ſine of the angle adjacent to it.

Fig. XV.

For in the rightangled ſpherical triangle C E F, as the ſine of the hypotenuſe C F to the radius, ſo is the ſine of the ſide F E to the ſine of the angle E C F[a]: i. e. in the rightangled ſpherical triangle A B C, as the coſine of the ſide A C to the radius, ſo is the coſine of the angle A B C[b], oppoſite to that ſide, to the ſine of the angle B C A[c], adjacent to it. The ſame will be demonſtrated of the ſide A B, if from C, as the pole, an arch of a great circle be deſcribed, on the ſide of A B.

[a] 18. 2.

[b] 4. cor. 9. def. 1.

[c] 4. or 6. cor. 9 def. 1.

Cor. In a quadrantal ſpherical triangle, as the coſine of either of the two angles adjacent to the quadrant, to the radius; ſo is the coſine of the ſide oppoſite to that angle, to the ſine of the ſide which is adjacent to it[d].

[d] prop. & 7. 2.

Theor.

PROPOSITION XXIII.

In a rightangled ſpherical triangle, as the coſine of the hypotenuſe to the radius, ſo is the cotangent of one of the two angles, excluding the right, to the tangent of the other angle.

Fig. XV.

For in the rightangled ſpherical triangle CEF, as the ſine of one of the ſides, C E, to the radius, ſo is the tangent of the other ſide F E to the tangent of the angle E C F[a]: i. e. in the rightangled ſpherical triangle A B C, as the coſine of the hypotenuſe B C to the radius, ſo is the cotangent of the angle A B C[b] to the tangent of the angle B C A[c]; and ſo is the cotangent of the angle B C A to the tangent of the angle A B C[d].

Cor. In a quadrantal ſpherical triangle, as the coſine of the angle ſubtended by the quadrant, to the radius; ſo is the cotangent of one of the two ſides, excluding the quadrant, to the tangent of the other ſide[e].

PROPOSITION XXIV.

Theor.

In every ſpherical triangle, the ſines of the ſides are proportional to the ſines of their oppoſite angles.

In rightangled and quadrantal ſpherical triangles this is manifeſt: for in a rightangled ſpherical triangle, as the ſine of the hypotenuſe to the radius, i. e. to the ſine of the right angle[a] which is oppoſite; ſo is the ſine of either of the two ſides about the right angle to the ſine of the angle oppoſite to it[b]: and alſo in a quadrantal ſpherical triangle, as the ſine of the angle, ſubtended by the quadrant, to the radius, i. e. to the ſine of that quadrant[a]; ſo is the ſine of either of the two angles adjacent to the quadrant, to the ſine of the ſide oppoſite to it[c].

But let A B C be an obliqueangled ſpherical triangle. I ſay, that as the ſine of the ſide B A to the ſine of the ſide A C, ſo is the ſine of the angle C to the ſine of the angle B.

Figure XI. XII.

From A, the point of meeting of the ſides B A, A C, draw an arch of a great circle, A D, perpendicular to the baſe B C, produced if neceſſary. And becauſe A B D is a rightangled ſpherical triangle, as the ſine of the ſide B A to the radius, ſo is the ſine of the perpendicular A D to the ſine of the angle B[b]. For the ſame reaſon, as the radius to the ſine of the ſide A C, ſo is the ſine of the angle A C D, or A C B[d], to the ſine of the perpendicular A D: Therefore, ex aequali perturbatè, as the ſine of the ſide A B to the ſine of the ſide A C, ſo is the ſine of the angle C to the ſine of the angle B[e].

PROPO-

[a] 19. 2.

[b] 4. cor. 9. def. 1.

[c] 4. or 6. cor. 9. def. 1.

[d] cor. 1. 1.

[e] prop. & 7. 2.

[a] 2 cor. 4. def. 1.

[b] 18. 2.

[c] cor. 18. 2.

[d] 6. cor. 9. def. 1.

[e] El. 23. 5.

Theor.

PROPOSITION XXV.

Figure XI. XII.

In an obliqueangled ſpherical triangle ABC, if from the vertex A be drawn an arch of a great circle, AD, perpendicular to the baſe BC; the ſines of the ſegments of the baſe, BD, DC, are proportional to the cotangents of the angles at the baſe, B, C.

For as the ſine of the ſegment BD to the radius, ſo is the tangent of the perpendicular AD to the tangent of the angle B[a]; and as the radius to the ſine of the ſegment DC, ſo is the tangent of the angle ACD, or ACB[b], to the tangent of the perpendicular AD[a]: therefore, ex aequali perturbatè, the ſine of the ſegment BD is to the ſine of the ſegment DC, as the tangent of the angle C to the tangent of the angle B[c]; or as the cotangent of the angle B to the cotangent of the angle C[d].

[a] 19. 2.

[b] 6. cor. 9. def. 1.

[c] El. 33. 5.

[d] cor. 1. 1.

Cor. In like manner from the ſame prop. 19. & El. 22. 5. it may be demonſtrated, that the tangents of the ſegments of the baſe, BD, DC, are proportional to the tangents of the angles at the vertex, BAD, DAC.

Theor.

PROPOSITION XXVI.

Figure XI. XII.

The ſame things being ſuppoſed, the ſines of the angles at the vertex, BAD, DAC, are proportional to the coſines of the angles at the baſe, B, C.

For as the ſine of the angle BAD to the radius, ſo is the coſine of the angle B to the coſine of the perpendicular AD[a]; and as the radius to the ſine of the angle DAC, ſo is the coſine of the perpendicular AD to the coſine of the angle ACD[a], or ACB[b]: therefore, ex aequali, the ſine of the angle BAD is to the ſine of the angle DAC, as the coſine of the angle B to the coſine of the angle C[c].

[a] 22. 2.

[b] 6. cor. 9. def. 1.

[c] El. 22. 5.

PROPOSITION XXVII.

Theor.

Figure XI. XII.

The ſame things being ſuppoſed, the coſines of the ſegments of the baſe, BD, DC, are proportional to the coſines of the ſides BA, AC.

For as the coſine of the ſegment BD to the radius, ſo is the coſine of the ſide BA to the coſine of the perpendicular AD[a]; and as the radius to the coſine of the ſegment DC, ſo is the coſine of the perpendicular AD to the coſine of the ſide AC[a]: therefore, ex aequali, the coſine of the ſegment BD is to the coſine of the ſegment DC, as the coſine of the ſide BA to the coſine of the ſide AC[b].

[a] 20. 2.

[b] El. 22. 5.

PROPOSITION XXVIII.

Theor.

Figure XI. XII.

The ſame things being ſuppoſed, the coſines of the angles at the vertex, are proportional to the cotangents of the ſides BA, AC.

For as the coſine of the angle BAD to the radius, ſo is the cotangent of the ſide BA to the cotangent of the perpendicular AD[a]; and as the radius to the coſine of the angle DAC, ſo is the cotangent of the perpendicular AD to the cotangent of the ſide AC[a]: therefore, ex aequali, the coſine of the angle BAD is to the coſine of the angle DAC, as the cotangent of the ſide BA to the cotangent of the ſide AC[b].

[a] 21. 2.

[b] El. 22. 5.

PROPOSITION XXIX.

Theor.

Fig. XVI.

In every ſpherical triangle ABC, as the rectangle contained by the ſine of half the perimeter of the triangle, and the ſine of the exceſs of half the perimeter above the baſe B C, to the rectangle contained by the ſines of the exceſſes of half the perimeter above each of the two remaining ſides B A, A C; ſo is the ſquare of the radius to the ſquare of the tangent of half the angle B A C, contained by thoſe two ſides.

Let D be the center of the ſphere: join D A, and complete the circle B E A F; and from A, as the pole, at the diſtance of the chord of the arch A C, deſcribe the circumference of a circle, E C F, meeting the circumference of the circle A B in the points E, F: draw E F, meeting A D in G; and join C E, C F, C G. Alſo from B, as the pole, at the diſtance of the chord of the arch B C, deſcribe the circumference of a circle, H C K, meeting the circumference of the circle A B in the points H, K: draw H K, meeting E F in L; and join C L, E H, H F; and biſect E H, H F in the points M, N.

Becauſe A is the pole of the circle E C F; each of the arches A E, A F is equal to the arch A C[a]; and A D is the axis of the circle E C F[b], and therefore paſſes through its center, and is at right angles to it. Wherefore G is the center of that circle, and E C F is a ſemicircle; and A G is at right angles to each of the ſtrait lines E G, G C[c]. Becauſe then in the planes of the arches A B, A C, there are drawn the ſtrait lines E G, G C at right angles to A G, the common ſection of thoſe planes; the angle E G C therefore is the inclination of thoſe planes[d], or the ſpherical angle B A C[e]; and the angle E F C is its half[f]. And becauſe B is the pole of the circle H C K; each of the arches B H, B K is equal to the arch B C[a]; and the circle A B H is perpendicular to the circle H C K[g], and converſely, the circle H C K is perpendicular to the circle A B H. But alſo the circle E C F is perpendicular to the ſame circle A B H[g], becauſe the latter paſſes through the pole of the former: therefore the common ſection of the circles H C K, E C F, i. e. the ſtrait line C L, is perpendicular to the circle A B H[h]; and therefore to the ſtrait line E F[c].

And becauſe the arch A F is equal to the arch A C, and the arch B H to the arch B C; the arch F B H then is the whole perimeter of the triangle A B C, and F N the ſine of half that perimeter[k]. And the

[a] 4. cor. 3. def. 2.
[b] 5. cor. 1. 2.
[c] El. 3. def. 11.
[d] El. 6. def. 11.
[e] 7. def. 2.
[f] El. 20. 3.
[g] 2. 2.
[h] El. 19. 11.
[k] 1. cor. 4. def. 1.

arch F K is the exceſs of the whole perimeter above twice the baſe B C: wherefore the ſine of half the arch F K, or the ſine of the angle F H K[f], is the ſine of the exceſs of half the perimeter above the baſe B C[l]. Alſo, the arch E H is the exceſs of the whole perimeter above the arch E F, i. e. above twice the ſide A C: and ſo E M is the ſine of the exceſs of half the perimeter above the ſide A C[k]. And becauſe the exceſs of the whole perimeter above twice the ſide A B, is the ſame as the exceſs of the two ſides B C, C A above A B; and the exceſs of the ſides B C, C A, or of the arches B K, A E, above A B is the arch E A K: therefore the ſine of half the arch E A K, or the ſine of the angle E H K[f], is equal to the ſine of the exceſs of half the perimeter above the ſide A B[l].

Then in the triangles F H L, E H L, F H is to F L, as the ſine of the angle F L H to the ſine of the angle F H L[m]; and E H is to E L, as the ſine of the angle E L H to the ſine of the angle E H L[m]: wherefore the rectangles contained by the extremes have to one another the ſame ratio, as the rectangles contained by the means[n]. Therefore the rectangle contained by F H and the ſine of the angle F H L, is to the rectangle contained by E H and the ſine of the angle E H L; or their halves, i. e. the rectangle contained by F N and the ſine of the angle F H L, is to the rectangle contained by E M and the ſine of the angle E H L[o]; as the rectangle contained by F L and the ſine of the angle F L H, to the rectangle contained by E L and the ſine of the angle E L H; i. e. (becauſe the ſines of the angles F L H, E L H are the ſame[p]), as F L to L E[q]. But as F L to L E, ſo is the ſquare of F L to the ſquare of L C[r]; for the triangle E C F is rightangled[s], and C L perpendicular to E F, as has been ſhewn: and the ſquare of F L is to the ſquare of L C, as the ſquare of the radius to the ſquare of the tangent of the angle E F C[t]: Therefore as the rectangle contained by F N and the ſine of the angle F H L, (i. e. the rectangle contained by the ſine of half the perimeter of the triangle, and the ſine of its exceſs above the baſe B C), to the rectangle contained by E M and the ſine of the angle E H L, (i. e. to the rectangle contained by the ſines of the exceſſes of half the perimeter above the ſides A C, A B); ſo is the ſquare of the radius to the ſquare of the tangent of the angle E F C, i. e. to the ſquare of the tangent of half the ſpherical B A C. Q. E. D.

Cor. 1. Put half the perimeter of the triangle A B C = $\frac{1}{2}$ P; and it

[f] El. 20. 3.
[l] 4. cor. 9. def. 1.
[k] 1 cor. 4. def. 1.
[m] 12. 1.
[n] El. 16. 6. & 7. 5.
[o] El. 1. 6. & 15. 5.
[p] 6. cor. 9. def. 1.
[q] El. 1. 6.
[r] El. cor. 8. 6. & 2. cor. 20. 6.
[s] El. 31. 3.
[t] 10. 1. & El. 22. 6.

will be, Sin. $\overline{\frac{1}{2}P - AB}$ × ſin. $\overline{\frac{1}{2}P - AC}$: ſin. $\frac{1}{2}P$ × ſin. $\overline{\frac{1}{2}P - BC}$ :: Rad.^{2} : cotang.^{2} $\frac{1}{2}A$; which may be demonſtrated in like manner from the triangles EHL, FHL, ECL; or by means of the cor. prop. 1.

Cor. 2. Sin. AB × ſin. AC : ſin. $\overline{\frac{1}{2}P - AB}$ × ſin. $\overline{\frac{1}{2}P - AC}$ :: Rad.^{2} : ſin.^{2} $\frac{1}{2}A$. Alſo, Sin. AB × ſin. AC : ſin. $\frac{1}{2}P$ × ſin. $\overline{\frac{1}{2}P - BC}$ :: Rad.^{2} : cos.^{2} $\frac{1}{2}A$.

Join BD meeting HK therefore BD is at right angles to HK[b]: but alſo AD is at right angles to EF[b]: wherefore the remaining angles D, L of the quadrilateral DL are equal to two right angles[u]: whence the angle BDA is equal to the angle ELH[v]; and the ſine of the angle BDA, or of the arch AB, is equal to the ſine of the angle ELH[l], and alſo to the ſine of its ſupplement[p], which is the angle FLH. Alſo, the ſine of the angle EHF is equal to the ſine of the arch AE[f], or AC; and the ſine of the angle EFH, or the ſine of half the arch EH[f], is the ſine of the exceſs of half the perimeter above AC; and the ſine of the angle FEH is the ſine of half the arch FH[f], or of half the arch FBH[p], i. e. of half the perimeter. Becauſe then in the triangles FHL, EHF, ſin. ELH : ſin. EHL :: EH : EL[m]; and ſin. EHF : ſin. EFH :: EF : EH[m]; therefore (ſin. ELH × ſin. EHF =) ſin. AB × ſin. AC : (ſin. EHL × ſin. EFH =) ſin. $\overline{\frac{1}{2}P - AB}$ × ſin. $\overline{\frac{1}{2}P - AC}$ :: EH × EF : EL × EH[w] :: EF : EL[q] :: EF^2 : EC^2[r] :: Rad.^{2} : ſin.^{2} $\frac{1}{2}A$[t]. Alſo, in the triangles FHL, EHF, ſin. FLH : ſin. FHL :: FH : FL[m]; and ſin. EHF : ſin. FEH :: EF : FH[m]; therefore (ſin. FLH × ſin. EHF =) ſin. A–B × ſin, AC : (ſin. FHL × ſin. FEH =) ſin. $\frac{1}{2}P$ × ſin. $\overline{\frac{1}{2}P - BC}$ :: FH × EF : FL × FH[w] :: EF : FL[q] :: EF^2 : FC^2[r] :: Rad.^{2} : cos^2. $\frac{1}{2}A$[t]. And the rectangle, ſin. $\overline{\frac{1}{2}P - AB}$ ×

ſin. $\overline{\frac{1}{2}P - AC}$ = ſin. $\frac{BC + AC - AB}{2}$ × ſin. $\frac{BC + AB - AC}{2}$;

and the rectangle, ſin. $\frac{1}{2}P$ × ſin. $\overline{\frac{1}{2}P - BC}$ = ſin. $\frac{AB + AC + BC}{2}$

× ſin. $\frac{AB + AC - BC}{2}$; as may appear from the common notions.

Cor. 3. Sin. AB × Sin. AC : ſin. $\overline{\frac{1}{2}P - AB}$ × ſin. $\overline{\frac{1}{2}P - AC}$ ::

2 rad.

[b] 5. cor. 1. 2.
[u] El. 1. cor. 32. 1.
[v] El. 13. 1.
[l] 4. cor. 9. def. 1.
[p] 6. cor. 9. def. 1.
[f] El. 20. 3.
[m] 12. 1.
[w] El. 23. 6.
[q] El 1. 6.
[r] El. cor. 8. 6. & 2. cor. 20. 6.
[t] 10. 1. & El. 22. 6.

2 rad. : vers. ſin. A. Alſo, Sin. A B × ſin. A C : ſin. ½ P × ſin. $\overline{\tfrac{1}{2}\,P - BC}$ :: 2 rad. : vers. ſin. ſuppl. A.

For it has been demonſtrated in the preceding cor. that theſe rectangles are to one another, as E F to E L, and as E F to F L; and E L, F L are the verſed ſines of the angle E G C, or A, and of its ſupplement, E F being the diameter. And each of theſe analogies may be ſplit into two, thus. Sin. A C : ſin. $\overline{\tfrac{1}{2}\,P - AB}$ :: ſin. $\overline{\tfrac{1}{2}\,P - AC}$: ſome ſine; and ſin. A B : that ſine :: 2 rad. : vers. ſin. A[x]. In like manner the ſecond analogy may be ſplit into two.

[x] El. F. 5. & 1. 6.

Cor. 4. From the prop. and the Elements it may be demonſtrated, that ſin. $\overline{\tfrac{1}{2}\,P - AC}$: ſin. $\overline{\tfrac{1}{2}\,P - BC}$ :: tang. ½ A : tang. ½ B.

Cor. 5. In an equilateral ſpherical triangle, as the ſine of one and a half of its ſide, to the ſine of half that ſide; ſo is the ſquare of the radius to the ſquare of the tangent of half its angle.

PROPOSITION XXX.

Theor.

Figure V.

In every ſpherical triangle ABC, as the rectangle contained by the coſine of half the ſum of its three angles, and the coſine of the difference between half that ſum and one of its angles, A, to the rectangle contained by the coſines of the differences between half the ſum of the three angles and each of the two remaining angles, B, C; ſo is the ſquare of the radius to the ſquare of the cotangent of half the ſide BC, adjacent to thoſe two remaining angles B, C.

Take the triangle DEF, ſupplemental of the triangle ABC. Therefore as the rectangle contained by the ſine of half the perimeter of the triangle DEF, and the ſine of its exceſs above the baſe EF, to the rectangle contained by the ſines of the exceſſes of half the perimeter above each of the two remaining ſides DF, DE; ſo is the ſquare of the radius to the ſquare of the tangent of half the angle D[a]. And becauſe as half the perimeter of the triangle D E F, to the circumference; ſo is half of

[a] 29. 2.

the ſupplements of the angles A, B, C, or the exceſs of three right angles above half the ſum of the angles A, B, C, to four right angles[b]: but this exceſs, or its ſupplement, is the complement of half the ſum of thoſe angles[c]: therefore the ſine of half the perimeter of the triangle D E F is equal to the coſine of half the ſum of the three angles A, B, C[d]. Alſo, becauſe as the exceſs of half the perimeter of the triangle DEF above the baſe EF, to the circumference; ſo is the exceſs of three right angles above half the ſum of the angles A, B, C, and above the ſupplement of the angle A, to four right angles[b]: and the exceſs of three right angles above half the ſum of the angles A, B, C, and above the ſupplement of A, is equal to the exceſs of the angle A together with one right angle, above half the ſum of the angles A, B, C: therefore the ſine of the exceſs of half the perimeter of the triangle DEF above the baſe EF, is equal to the ſine of one right angle together with the difference between half the ſum of the angles A, B, C, and the angle A[d]; i. e. it is equal to the coſine of the difference between half the ſum of the angles A, B, C, and the angle A. In like manner it may be demonſtrated, that the ſine of the exceſs of half the perimeter of the triangle DEF above the ſide DF, is equal to the coſine of the difference between half the ſum of the angles A, B, C, and the angle B; and the ſine of the exceſs of half the perimeter of the triangle DEF above the ſide DE, is equal to the coſine of the difference between half the ſum of the angles A, B, C, and the angle C. And it is manifeſt, that the tangent of half the angle D is equal to the cotangent of half the ſide BC[d]. Therefore the prop. is evident.

b 7. 2.
c 3. def. 1.
d 4. cor. 9. def. 1.

Cor. 1. Put half the ſum of the three angles A, B, C of the triangle ABC, $= \frac{1}{2}S$; and it will be, Cos. $\overline{\frac{1}{2}S - B} \times$ cos. $\overline{\frac{1}{2}S - C}$: cos. $\frac{1}{2}S \times$ cos. $\overline{\frac{1}{2}S - A}$ :: Rad.^{2} : tang.^{2} $\frac{1}{2}$ BC[e].

e 1. cor. 29. 2. & 7. 2.

Cor. 2. Sin. B × ſin. C : cos. $\overline{\frac{1}{2}S - B} \times$ cos. $\overline{\frac{1}{2}S - C}$ :: Rad.^{2} : cos.^{2} $\frac{1}{2}$ BC. Alſo, Sin. B × ſin. C : cos. $\frac{1}{2}S \times$ cos. $\overline{\frac{1}{2}S - A}$ :: Rad.^{2} : ſin.^{2} $\frac{1}{2}$ BC[f]. And the rectangle, Cos. $\overline{\frac{1}{2}S - B} \times$ cos. $\overline{\frac{1}{2}S - C} =$ cos. $\frac{A + C - B}{2} \times$ cos. $\frac{A + B - C}{2}$; and the rectangle, Cos. $\frac{1}{2}S \times$ cos. $\overline{\frac{1}{2}S - A} =$ Cos. $\frac{B + C + A}{2} \times$ cos. $\frac{B + C - A}{2}$; as may appear from the common notions.

f 2. cor. 29. 2. & 7. 2.

COR. 3. Sin B × ſin. C : cos. $\overline{\frac{1}{2} S - B}$ × cos. $\overline{\frac{1}{2} S - C}$ :: 2 rad. : vers. ſin. ſuppl. B C. Alſo, Sin. B × ſin. C : cos. $\frac{1}{2}$ S × cos. $\overline{\frac{1}{2} S - A}$ :: 2 rad. : vers. ſin. B C[g]. And each of theſe analogies may be ſplit into two, thus. Sin. C : cos. $\overline{\frac{1}{2} S - B}$ :: cos. $\overline{\frac{1}{2} S - C}$: ſome ſine; and Sin. B : that ſine :: 2 rad. : vers. ſin. ſuppl. B C[h]. And in like manner the ſecond analogy may be ſplit into two.

COR. 4. Cos. $\overline{\frac{1}{2} S - A}$: cos. $\overline{\frac{1}{2} S - B}$ :: (cot. $\frac{1}{2}$ A C : cot. $\frac{1}{2}$ B C[k] ::) tang. $\frac{1}{2}$ B C : tang. $\frac{1}{2}$ A C[l].

Cor. 5. In an equiangular ſpherical triangle, as the coſine of one and a half of its angle, to the coſine of half that angle; ſo is the ſquare of the radius to the ſquare of the cotangent of half its ſide.

Of Circular Elements.

DEFINITION.

IN a rightangled, or quadrantal, ſpherical triangle, the two elements adjacent to the right angle, or to the quadrant, and the complements of the three remaining elements, are called *the circular elements* of thoſe triangles: but the right angle and the quadrant are not reckoned among them, and are left alone. Of theſe five whichſoever three be taken, yet two of them cannot but be contiguous one to the other, and the third is either contiguous to theſe two, or not. In the firſt caſe, the element lying between the other two, is called *the middle element*, and the other two are called *adjacent extremes*: in the ſecond caſe, the element which is not contiguous to either of the other two, is called *the middle element*, and the other two *oppoſite extremes*.

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[g] 3. cor. 29. 2. & 7. 2.

[h] El. F. 5. & 1. 6.

[k] 4. cor. 29. 2. & 7. 2.

[l] cor. 1. 1.

THEOREM I.

In the circular elements, the coſines of the oppoſite extremes, and the tangents of the adjacent, are reciprocally proportional to the radius and the ſine of the middle element.

Fig. XV. For in the rightangled ſpherical triangle A B C, let firſt, either of the two ſides, as A B, be the middle element: wherefore the complement of the hypotenuſe B C and the complement of the angle C are the oppoſite extremes. But as the ſine of the hypotenuſe B C, i. e. the coſine of its complement, to the radius; ſo is the ſine of the ſide A B, to the ſine of the angle C[a], i. e. to the coſine of the complement of the angle C. Therefore the coſines of the oppoſite extremes, i. e. thoſe of the complements of the hypotenuſe B C and of the angle C, are reciprocally proportional to the radius and the ſine of the middle element which is A B.

The ſame thing being ſuppoſed, the ſide A C and the complement of the angle B are the adjacent extremes. And the tangent of the ſide A C is to the tangent of the angle B, as the ſine of the ſide A B to the radius[b]; or the tangent of the ſide A C is to the radius, as the ſine of the ſide A B to the cotangent of the angle B[c], i. e. to the tangent of the complement of the angle B. Therefore the tangents of the adjacent extremes, viz. thoſe of the ſide A C and of the complement of the angle B, are reciprocally proportional to the radius and the ſine of the middle element, viz. of A B.

Secondly, let the complement of the hypotenuſe B C be the middle element: wherefore the ſides A B, A C are the oppoſite extremes. And as the coſine of the ſide A C to the radius, ſo is the coſine of the hypotenuſe B C, i. e. ſine of the complement of the hypotenuſe B C, to the coſine of the ſide A B[d].

The ſame thing being ſuppoſed, the complements of the angles B, C are the adjacent extremes. And the cotangent of the angle B is to the tangent of the angle C, as the coſine of the hypotenuſe B C to the radius[e]; or

[a] 18. 2.

[b] 19. 2.

[c] cor. 1. 1.

[d] 20. 2.

[e] 2. 32.

or the cotangent of the angle B, i. e. the tangent of the complement of the angle B, is to the radius; as the coſine of the hypotenuſe BC, or the ſine of the complement of the hypotenuſe BC, to the cotangent of the angle C[c], i. e. to the tangent of the complement of the angle C.

Thirdly, let the complement of either of the two angles, as that of B, be the middle element: therefore the ſide AC and the complement of the angle C are the oppoſite extremes. And as the coſine of the ſide AC to the radius; ſo is the coſine of the angle B, i. e. the ſine of the complement of the angle B, to the ſine of the angle C[f], or to the coſine of the complement of the angle C.

The ſame thing being ſuppoſed, the ſide AB and the complement of the hypotenuſe BC are the adjacent extremes. And the cotangent of the hypotenuſe BC is to the cotangent of the ſide AB, as the coſine of the angle B to the radius[g]; or the cotangent of the hypotenuſe BC, i. e. tangent of the complement of the hypotenuſe BC, is to the radius; as the coſine of the angle B, or ſine of the complement of the angle B, to the tangent of the ſide AB[c].

The theorem is alſo evident on a quadrantal ſpherical triangle, by means of the corr. prop. 18—23.

THEOREM II.

Figure XI. XII.

In an obliqueangled ſpherical triangle ABC, if from the vertex A be drawn an arch of a great circle, AD, perpendicular to the baſe BC; and the perpendicular be taken for one of the two oppoſite, or one of the two adjacent extremes, in each of the two rightangled triangles ABD, ACD; and the remaining oppoſite, or adjacent, extremes be taken in both thoſe rightangled triangles in the ſame order: Then the coſines of the remaining oppoſite extremes, and the tangents of the remaining adjacent extremes will be proportional to the ſines of their middle elements.

For, firſt, let AD be one of the oppoſite extremes; and the complements of the angles B, C be the middle elements: wherefore the complements

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[c] cor. 1. 1.

[f] 22. 2.

[g] 21. 2.

ments of the angles B A D, D A C are the remaining oppoſite extremes. And in the triangle A B D, as the coſine of the complement of the angle B A D to the radius, ſo is the ſine of the complement of the angle B to the coſine of A D[a]: alſo, in the triangle A C D, as the radius to the coſine of the complement of the angle D A C, ſo is the coſine of A D to the ſine of the complement of the angle C[a]: wherefore, ex aequali, coſines of the complements of the angles B A D, D A C, i. e. coſines of the remaining oppoſite extremes, are proportional to the ſines of their middle elements, viz. to thoſe of the complements of the angles B, C[b]. But if the complements of the ſides B A, A C be the middle elements; the ſegments of the baſe, B D, D C, are the remaining oppoſite extremes. And in the triangle A B D, as the coſine of B D to the radius, ſo is the ſine of the complement of A B to the coſine of A D[a]: alſo, in the triangle A C D, as the radius to the coſine of D C, ſo is the coſine of A D to the ſine of the complement of A C[a]: wherefore, ex aequali, coſines of the remaining oppoſite extremes, the ſegments B D, D C are proportional to the ſines of their middle elements, to thoſe of the complements of the ſides A B, A C[b].

Secondly, let A D be one of the adjacent extremes; and the ſegments of the baſe, B D, D C, be the middle elements: therefore the complements of the angles B, C are the remaining adjacent extremes. And in the triangle A B D, as the tangent of the complement of the angle B to the radius, ſo is the ſine of B D to the tangent of A D[a]: alſo, in the triangle A C D, as the radius to the tangent of the complement of the angle C, ſo is the tangent of A D to the ſine of D C[a]: therefore, ex aequali, the tangents of the complements of the angles B, C, i. e. tangents of the remaining adjacent extremes, are proportional to the ſines of their middle elements, the ſegments B D, D C[b]. But if the complements of the angles B A D, D A C be the middle elements; then the complements of the ſides B A, A C are the remaining adjacent extremes. And in the triangle A B D, as the tangent of the complement of A B to the radius, ſo is the ſine of the complement of the angle B A D to the tangent of A D[a]: alſo, in the triangle A C D, as the radius to the tangent of the complement of A C, ſo is the tangent of A D to the ſine of the complement of the angle D A C[a]: therefore, ex aequali, the tangents of the complements of the ſides B A, A C, i. e. thoſe of the remaining adja-

[a] 1. theor. 2.

[b] El. 22. 5.

cent extremes, are proportional to the ſines of their middle elements, viz. to thoſe of the complements of the angles B A D, D A C[b].

Cor. And if the perpendicular A D be taken for the middle element; the coſines of the oppoſite extremes, in the triangle A B D, are reciprocally proportional to the coſines of the oppoſite extremes, in the triangle A C D; and the tangents of the adjacent extremes in the firſt triangle, to the tangents of the adjacent extremes in the ſecond triangle.

SCHOLIUM

To the following Table of Solutions of Spherical Triangles.

In every ſpherical triangle, if of the ſix of its elements, which are the three ſides and the three angles, any three be given, the remaining three will alſo be found, by analogies depending on the foregoing principles. Yet thoſe analogies are not always neceſſary; viz.

1ſt. When a ſpherical triangle has two, or three, angles right: for the ſides oppoſite to thoſe angles are quadrants, prop. 14. and the third ſide is of the ſame number of degrees, minutes, &c. as the angle oppoſite to it; and converſely, the third angle is of the ſame number of degrees, &c. as the ſide oppoſite to it, 3. cor. 2. & prop. 5.

2dly. When a ſpherical triangle has two, or three, ſides quadrants: for then the angles oppoſite to thoſe quadrants are right, cor. 14. prop. and the third angle is of the ſame number of degrees, minutes, &c. as the ſide oppoſite to it; and converſely, the third ſide is of the ſame number of degrees, &c. as the angle oppoſite to it. 5. cor. 2. & prop. 5.

3dly. When a ſpherical triangle has one of its ſides a quadrant, and one of its angles right: for it has another ſide, which is a quadrant, prop. 15. or its cor. 2. and the angle oppoſite to that ſide is right. prop. 14. and the third ſide is of the ſame number of degrees, minutes, &c. as the angle oppoſite to it; and converſely, 4 & 6. cor. 2. & prop. 5. In theſe three caſes however, the third ſide cannot be found from any of the other elements given, unleſs from the third angle; nor the third angle, except from the third ſide given.

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[b] El. 22. 5.

Of the other ſpherical triangles, thoſe that have but one right angle, and none of their ſides a quadrant, are ſolved by means of the propp. 18—23. or for the greater eaſe of the memory, by means of the 1ſt. theor. of the circular elements, as is done in the firſt of the following tables of ſolutions; and the ſolutions are limited from propp. 14 and 15 & cor.

Spherical triangles, that have but one of their ſides a quadrant, and none of their angles right, are ſolved by means of the corr. prop. 18—23. and their ſolutions are limited from cor. 14. prop. and 4. cor. 15. But ſuch triangles are more eaſily ſolved by the 1ſt theor. of the circular elements. They may alſo be ſolved by means of their ſupplemental triangles, which are rightangled; and whoſe ſides are of the ſame number of degrees, minutes, &c. as the ſupplements of the angles which are oppoſite to them in the firſt triangles; and whoſe angles are of the ſame number of degrees, &c. as the ſupplements of the ſides oppoſite to them in the firſt triangles. Wherefore if the triangle which is ſupplemental of a quadrantal triangle be ſolved, the quadrantal triangle is ſolved alſo: and for that reaſon, ſuch triangles are not commonly ſet down in tables of ſolutions.

Of obliqueangled ſpherical triangles, ſuch as have their three ſides, or their three angles, equal to one another, are ſolved by 5. cor. 29. and 5. cor. 30.

Obliqueangled ſpherical triangles, which have two ſides, or two angles, equal, are ſolved more eaſily, if from the vertex of the third angle be drawn a perpendicular upon the third ſide, which will biſect both that angle and that ſide; as may appear from propp. 27, 28, & 25, 26. Wherefore it will be ſufficient to ſolve one of the rightangled triangles, which are on each ſide of the perpendicular: and the angles are at once limited from the ſides, as well as the ſides from the angles, by 2. cor. 12. prop.

Obliqueangled ſpherical triangles, which have two of their ſides equal together to a ſemicircumference, or two of their angles equal together to two right angles, are ſolved more eaſily, if one of thoſe two ſides, as alſo the third ſide, be produced, till they meet again. For then there will be adjacent to ſuch a triangle, another ſpherical triangle, whoſe two ſides, or two angles are equal: if therefore this latter triangle be ſolved, it is manifeſt from 1. cor. 4. def. that either of the forementioned triangles will be ſolved alſo.

In the ſolutions of obliqueangled ſpherical triangles, there might have been three caſes only; becauſe from theſe three, the remaining three may be ſolved, by means of ſupplemental triangles: however for the ſake of perſpicuity and ſhortneſs of ſolution, all ſix are generally exhibited.

Obliqueangled ſpherical triangles, in thoſe caſes, when the three ſides, or the three angles, are given, are ſolved by propp. 29. and 30. or their corr. And when two ſides and an angle oppoſite to one of them being given, the angle oppoſite to the other ſide is ſought; or two angles and a ſide oppoſite to one of them being given, the ſide oppoſite to the other angle is ſought; the ſolution is performed by prop. 24. In all other caſes, obliqueangled triangles are commonly ſolved by means of a perpendicular drawn from the vertex of one of their angles. But it is to be obſerved, that the perpendicular is to be drawn from the end of a given ſide, and oppoſite to a given angle, and ſo that in one of the two rightangled triangles adjacent to the perpendicular, there be two out of the three given elements. And when the three given elements are contiguous to one another, the perpendicular muſt beſides be drawn from the end of a ſide ſought, and ſo as to be oppoſite to an angle ſought, or to its ſupplement. Then the triangle is commonly ſolved by means of one of the rightangled triangles which are adjacent to the perpendicular, and propp. 25—28. But inſtead of propp. 25—28. the 2d. theor. of the circular elements may, for the greater eaſe of the memory, be uſed. In that caſe the perpendicular is always to be taken for one of the two oppoſite, or one of the two adjacent, extremes: and the remaining oppoſite, or adjacent, extremes, as alſo the middle elements, will be the two angles, or the two ſides, of the obliqueangled triangle, and alſo the two angles at the vertex, or the two ſegments of the baſe, one of which has already been found from the rightangled triangle; and whoſe ſum, or difference is equal to the angle or the baſe given or ſought. Both theſe methods of ſolution are ſet down in the ſecond of the following tables. Solutions of obliqueangled triangles are limited from prop. 16. & cor. and prop. 17. as alſo from propp. 8, 9, 11, 12. & corr. 13. & corr. But the caſes of obliqueangled ſpherical triangles, wherein a perpendicular is uſed, may, without ſuch a perpendicular, be both ſolved and limited by means of ſome of the theorems contained in the appendix after the tables.

Spherical triangles, in which two ſides and an angle oppoſite to one of them are given, whether the other of them be a quadrant, or not; or two angles and a ſide oppoſite to one of them are given, whether the other of them be right, or not; have a double ſolution of their other elements. *Figure* V. For if one of the two given ſides be a quadrant, or one of the two given angles be right; two ſpherical triangles ABC, ABD, in which two ſides BC, BD are quadrants, or the two angles at B right ones, alſo the ſide AB common, and the angle C, D equal, may not have the other ſides and angles equal, but ſupplements of one another. The ſame holds in *Fig.* XVII. obliqueangled ſpherical triangles. For if there be drawn two arches of great circles, CAD, CBD, not at right angles to one another, meeting in the points C, D; and the arch CAD be divided in A into unequal ſegments CA, AD; and from A be drawn an arch of a great circle, AB, perpendicular to the arch CBD; and from the point B be taken on each ſide equal ſegments BE, BF; and arches of great circles, AE, AF, be drawn: Then ſhall AE be equal to AF, as appears from prop. 20 & 15. and the angle BEA equal to the angle BFA, as appears from prop. 19 & 14. wherefore alſo the angle AEC is equal to the angle AFD. Therefore there are two ſpherical triangles ACE, ACF, which have the ſides AE, AF equal; and the ſide CA, as well as the angle C common: but the reſt is not equal. And the ſame is in the triangles ADE, ADF. Alſo, in the ſpherical triangles ACF, ADE, the angles C and CFA are equal to the angles D and DEA, each to each; and the ſide AF is equal to the ſide AE: but the reſt is not equal. And the ſame is in the triangles ACE, ADF. Therefore in theſe caſes the ſolution of the remaining elements of the triangles is double: and the elements of each ſolution are conformed together by cor. 16. and prop. 17. But in all other caſes, ſpherical triangles have a ſingle ſolution, as may be demonſtrated from propp. 18—30. and others.

TABLE

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Fig. XV.

TABLE OF SOLUTIONS

Of Rightangled Spherical Triangles.

	Given.	Sought.	The Solution.
1.	B C, C A; hypotenuſe and a ſide.	the other ſide, A B.	Cos. C A : Rad. :: ſin. compl. B C : cos. A B. 1. theor. i. e. Cos. C A : Rad. :: cos. B C : cos. A B (prop. 20.) of the ſame, or of different, affection with C A, according as the hypotenuſe B C is leſſer, or greater, than a quadrant. 2. cor. 15.
		the angle B. oppoſite to C A.	Cos. compl. B C : Rad. :: ſin. A C : cos. compl. B. 1. theor. i. e. ſin. B C : Rad. :: ſin. A C : ſin. B (prop. 18.) of the ſame affection with the ſide A C. prop. 14.
		the angle C, adjacent to C A,	Rad. : tang. compl. B C :: tang. C A : ſin. com. C. 1. theor. Or Cot. C A : cot. B C :: Rad. : cos. C (prop. 21). of the ſame, or of different, affection with A C, according as the hypotenuſe B C is leſſer, or greater, than a quadrant. 3. cor. 15.
2.	B C, B; hypotenuſe and an angle.	the oppoſite ſide, A C.	Rad. : cos. com. B C :: cos. com. B : ſin. A C. 1. theor. i. e. Rad. : ſin. B C :: ſin. B : ſin. A C (prop. 18). of the ſame affection with the angle B. prop. 14.
		the adjacent ſide, A B.	Tang. com. B C : Rad. :: ſin. com. B : tang. A B. 1. theor. Or Cos. B : Rad. :: cot. A B (prop. 21). of the ſame, or of different, affection with the angle B, according to the hypotenuſe B C is leſſer, or greater, than a quadrant. 3. cor. 15.

Fig. XV.

	Given.	Sought.	The Solution.
		the other angle, C.	Tang. com. B : Rad. :: ſin. com. BC : tang. com. C. 1. theor. Or Cos. BC : Rad. :: cot. B : tang. C (prop. 23), of the ſame, or of different, affection with the angle B, according as the hypotenuſe BC is leſſer, or greater, than a quadrant. 2. cor. 15.
3.	BA, AC; two ſides.	the hypotenuſe BC.	Rad. : cos. BA :: cos. AC : ſin. com. BC. 1. theor. i. e. Rad. : cos. BA :: cos. AC : cos. BC (prop. 20), leſſer, or greater, than a quadrant, according as the two ſides are of the ſame, or of different affection. prop. 15.
		the angle B.	Tang. AC : Rad. :: ſin. BA : tang. com. B. 1. theor. Or Sin. BA : Rad :: tang. AC : tang. B (prop. 19), of the ſame affection with AC. prop. 14.
		the angle C.	Tang. BA : Rad. :: ſin. AC : tang. com. C. 1. theor. Or Sin. AC : Rad. :: tang. BA : tang. C. (prop. 19), of the ſame affection with AB. prop. 14.
4.	AC, B; a ſide, and the angle oppoſite.	the other angle, C.	Cos. AC : Rad. :: ſin. com. B : cos. com. C. 1. theor. i. e. Cos. AC : Rad. :: cos. B : ſin. C (prop. 22); but whether leſſer, or greater, than a right angle, is uncertain; unleſs it may be determined from 3. cor. 13.
		the other ſide, AB.	Rad. : tang. AC :: tang. com. B : ſin. AB. 1. theor. Or Tang. B : tang. AC :: Rad. : ſin. AB. (prop 19); but whether leſſer, or greater, than a quadrant, is uncertain; unleſs it may be determined from the angle C.
		the hypotenuſe BC.	Cos. com. B : Rad. :: ſin. AC : cos. com. BC. 1. theor. i. e. Sin. B : ſin. AC :: Rad. : ſin. BC (prop. 18); but whether leſſer, or greater, than

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Fig. XV.

	Given.	Sought.	The Solution.
		the hypotenuſe B C.	than a quadrant, is uncertain; unleſs it may be determined from the angle C.
5.	A B, B; a ſide, and the angle adjacent.	the hypotenuſe, B C.	Tang. A B : Rad. :: ſin. com. B : tang. com. B C. 1 theor. Or Rad. : cos. B :: cot. A B : cot. B C (prop. 21), leſſer, or greater, than a quadrant, according as the ſide is of the ſame, or of different, affection with the angle. 3. cor. 15.
		the other ſide, A C.	Tang. com. B : Rad. :: ſin. A B : tang. A C. 1. theor. Or Rad. : ſin. A B :: tang. B : tang. A C (prop. 19), of the ſame affection with the angle B. prop. 14.
		the other angle, C.	Rad. : cos. A B :: cos. com. B : ſin. com. C. 1. theor. i. e. Rad. : cos. A B :: ſin. B : cos. C (prop. 22), of the ſame affection with A B. prop. 14.
6.	B C; two angles	the hypotenuſe, B C.	Rad. : tang com. B :: tang. com. C : ſin. com. B C. 1. theor. Or Tang. C : cot. B :: Rad. : cos. B C (prop. 23), leſſer, or greater, than a quadrant, according as the angles are of the ſame, or of different, affection. 1. cor. 15.
		the ſide A C.	Cos. com. C : Rad. :: ſin. com. B : cos. A C. 1. theor. i. e. Sin. C : cos. B. :: Rad. : cos. A C (prop. 22) of the ſame affection with the angle B. prop. 14.
		the ſide A B.	Cos. com. B : Rad. :: ſin. com. C : cos. A B. 1. theor. i. e. Sin. B : cos. C :: Rad. : cos. A B (prop. 22), of the ſame affection with the angle C. prop. 14.

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Figure XI.
XII.

TABLE OF SOLUTIONS

Of Obliqueangled Spherical Triangles.

	Given.	Sought.	The Solution.
1.	BA, AC, and B; two ſides, and one of the oppoſite angles.	the other oppoſite angle C.	Sin. AC : ſin. AB :: ſin. B : ſin. C (prop. 24); but whether leſſer, or greater, than a right angle, is uncertain; unleſs it may be determined by propp. 11. and 12.
		the third ſide, BC.	From A to BC draw the perpendicular AD. Then 1ſt. Tang. com. AB : Rad. :: ſin. com. B : tang. BD. 1. theor. Or cos. B : Rad. :: cot. BA : cot. BD (prop. 21), limited by 3. cor. 15. 2dly. Sin. com. BA : ſin. com. AC :: cos. BD : cos. DC. 2. theor. i. e. cos. BA : cos. AC :: cos. BD : cos. DC (prop. 27), of the ſame, or of different, affection with BD, according as the ſides BA, AC are of the ſame, or of different, affection. prop. 17. 3dly. BC may be the ſum, as well as the difference, of the two ſegments BD, DC: only it cannot be their ſum, when that ſum is not leſſer than a ſemicircumference; nor can it be their difference, when BD is leſſer than DC: otherwiſe the ſolution is double, except any thing beſide can be determined from propp. 8 and 9. 4thly. When the ſolution is double; according as the angle C is taken of the ſame, or of different, affection with the angle B; BC is the ſum, or the difference, of the two ſegments BD, DC. cor. 16.
		the contained angle, A.	From A to BC draw the perpendicular AD. Then 1ſt. Tang. com. B : Rad. :: ſin. com. BA :

Figure XI.
XII.

Given.	Sought.	The Solution.
		BA : tang. com. BAD. 1. theor. Or cos. BA : Rad. :: cot. B : tang. BAD (prop. 23), limited by 2. cor. 15. 2dly. Tang. com. BA : tang. com. AC :: ſin. com. BAD : ſin. com. DAC. 2. theor. i. e. Cot. BA : cot. AC :: cos. BAD : cos. DAC (prop. 28), of the ſame, or of different, affection with the angle BAD, according as the ſides BA, AC are of the ſame, or of different, affection, prop. 17. 3dly. the angle A may be the ſum, as well as the difference, of the two angles BAD, DAC: only it cannot be their ſum, when that ſum is not leſſer than two right angles; nor can it be their difference, when the angle BAD is leſſer than the angle DAC: otherwiſe the ſolution is double, except any thing beſide may be determined by means of the ſide BC. When the ſolution is double; according as the angle C is taken of the ſame, or of different, affection with the angle B; the angle A is the ſum, or the difference, of the two angles BAD, DAC. cor. 16.
2.	the other oppoſite ſide, AC.	Sin. C : ſin. B :: ſin. AB : ſin. AC (prop. 24); but whether leſſer, or greater, than a quadrant, is uncertain; unleſs it may be determined by propp. 11 & 12.
	the third angle A.	From A to BC draw the perpendicular AD; which will fall within, or without, the triangle, according as the angles B, C are of the ſame, or of different, affection. cor. 16. Then 1ſt. Tang. com. B : Rad. :: ſin. com. BA : tang. com. BAD. 1. theor. Or cos. BA : Rad. :: cot. B : tang. BAD (prop. 23), limited

Figure XI. XII.

Given.	Sought.	Solution.
B, C, and AB; two angles, and one of the oppoſite ſides.		mited by 2. cor. 15. 2dly. Sin. com. B : ſin. com. C :: cos. com. BAD : cos. com. DAC. 2. theor. i. e. Cos. B : cos. C :: ſin. BAD : ſin. DAC (prop. 26), undetermined. 3dly. The angle A will be the ſum, or the difference, of the two angles BAD, DAC, according as the perpendicular falls within, or without, the triangle. When the perpendicular falls within, the angle A may be the ſum both of the angles BAD and DAC, and alſo of the angle BAD and the ſupplement of DAC: unleſs indeed this latter ſum is not leſſer than two right angles. And when the perpendicular falls without, the angle A may be the difference both of the angles BAD and DAC, and alſo of the angle BAD and the ſupplement of DAC: unleſs indeed the ſupplement of DAC is greater than the angle BAD. The ſolution therefore is double, except any thing beſide may be determined from 2. cor. 13. 4thly. When the ſolution is double; according as AC be taken of the ſame, or of different, affection with AB; the angle DAC alſo is of the ſame, or of different, affection with the angle BAD. prop. 17.
	the ſide adjacent to the angles, B, C.	From A to BC draw the perpendicular AD; which will fall within, or without, the triangle, according as the angles B, C are of the ſame, or of different, affection. cor. 16. Then 1ſt. Tang. com. AB : Rad. :: ſin. com. B : tang. BD. 1. theor. Or cos. B : Rad. :: cot. AB : cot. BD (prop. 21), limited by 3. cor. 15. 2dly. Tang. com. B : tang. com. C :: ſin. BD : ſin. DC. 2. theor. i. e. Cot. B : cot. C :: ſin. BD : ſin. DC (prop. 25), undetermined.

	Given.	Sought.	Solution.
			determined. 3dly. BC will be the ſum, or the difference, of the two ſegments BD, DC, according as the perpendicular falls within, or without, the triangle. When the perpendicular falls within, BC may be the ſum both of BD and DC, and alſo of BD and the ſupplement of DC: unleſs indeed this latter ſum is not leſſer than a ſemicircumference. And when the perpendicular falls without, BC may be the difference both of BD and DC, and alſo of BD and the ſupplement of DC: unleſs indeed the ſupplement of DC is greater than BD. The ſolution therefore is double, except any thing beſide may be determined by means of the angle A. 4thly. When the ſolution is double; according as AC is taken of the ſame, or of different, affection with AB; the ſegment DC alſo is of the ſame, or of different, affection with BD. prop. 17.
3.	AB, BC, and B; two ſides, and the contained angle.	one of the oppoſite angles, C.	From A to BC draw the perpendicular AD. Then 1ſt. Tang. com. AB : Rad. :: ſin. com. B : tang. BD. 1. theor. Or Cos. B : Rad. :: cot. AB : cot. BD (prop. 21), limited by 3. cor. 15. 2dly. If BD be equal to BC, the angle C is right: but if not, take their difference DC. 3dly. Sin. BD : ſin. DC :: tang. com. B : tang. com. C. 2. theor. i. e. Sin. BD : ſin. DC :: cot. B : cot. C prop. 25), of the ſame, or of different, affection with the angle B, according as BD is leſſer, or greater, than BC. prop. 16.
		the other oppoſite angle, A.	From C to AB draw a perpendicular; and proceed, as in finding of the angle C.

To

Figure XI. XII.

	Given.	Sought.	Solution.
		The third ſide, A C.	To either of the given ſides, as B C, draw, from the vertex of the oppoſite angle A, the perpendicular A D: and 1ſt. find B D, as before. 2dly. If B D be equal to B C, the triangle is rightangled, and the angle C is right: wherefore proceed, as in rightangled triangles: but if B D be not equal to B C, take their difference D C. 3dly. Cos. B D : cos. D C :: ſin. com. B A : ſin. com. A C. 2. theor. i. e. Cos. B D : cos. D C :: cos. B A : cos. A C (prop. 27), of the ſame, or of different, affection with B A; according as the two ſegments B D, D C are of the ſame, or of different, affection. prop. 17. The ſame may be found from either of the angles C, A, by prop. 24. provided the affection of the ſegments be known.
4	A, B, and A B, two angles and the ſide adjacent to them.	one of the oppoſite ſides, A C.	From A to B C draw the perpendicular A D. Then, 1ſt. Tang. com. B : Rad. :: ſin. com. B A : tang. com. B A D. 1. theor. Or Cos. B A: Rad. :: cot. B : tang. B A D (prop. 23), limited by 2. cor. 15. 2dly. If the angle B A D be equal to the angle B A C, the triangle is rightangled, and the angle C is right: therefore proceed, as in rightangled triangles. But if the angles B A D, B A C be not equal; take their difference, the angle D A C. 3dly. Sin. com. B A D : ſin. com. D A C :: tang. com. B A : tang com. A C. 2. theor. i. e. Cos. B A D : cos. D A C :: cot. B A : cot. A C (prop. 28), of the ſame, or of different, affection with B A, according as the angles B A D, D A C are of the ſame, or of different, affection. prop. 17.
		the other oppoſite ſide, B C.	From B to A C draw a perpendicular; and proceed, as in finding of the ſide A C.

From

Book II.
Figure XI.
XII.

	Given.	Sought.	Solution.
		The third angle, C.	From the vertex of one of the given angles, as A, draw, to the oppoſite ſide B C, the perpendicular A D: and 1ſt. find the angle B A D, as before. 2dly. If the angle B A D be equal to B A C, the angle C is right: but if not, take the angle D A C, which is the difference of the angles B A D, B A C. 3dly. Cos. com. B A D : cos. com. D A C :: ſin. com. B : ſin. com. C. 2. theor. i. e. Sin. B A D : ſin. D A C :: cos. B : cos. C (prop. 26), of the ſame, or of different, affection with the angle B, according as the angle B A D is leſſer, or greater, than the angle B A C, prop. 16. The ſame may be found from either of the ſides A C, B C, by prop. 24. provided the affection of the angles at the vertex be known.
5.	A B, A C, B C; the three ſides.	The angles;	Put half the perimeter of the triangle, $\frac{AB + AC + BC}{2} = \tfrac{1}{2} P$. Then
		A,	Sin. $\tfrac{1}{2}$ P × ſin. $\overline{\tfrac{1}{2} P - BC}$: ſin. $\overline{\tfrac{1}{2} P - AB}$ × ſin. $\overline{\tfrac{1}{2} P - AC}$ :: Rad.^{2} : tang.^{2} $\tfrac{1}{2}$ A. prop. 29.
		B,	Sin. $\tfrac{1}{2}$ P × ſin. $\overline{\tfrac{1}{2} P - AC}$: ſin. $\overline{\tfrac{1}{2} P - AB}$ × ſin. $\overline{\tfrac{1}{2} P - BC}$ :: Rad.^{2} : tang.^{2} $\tfrac{1}{2}$ B prop. 29. Or the angle B may be found from the angle A, thus. Sin. $\overline{\tfrac{1}{2} P - AC}$: ſin. $\overline{\tfrac{1}{2} P - BC}$ :: tang. $\tfrac{1}{2}$ A : tang. $\tfrac{1}{2}$ B. 4. cor. 29.
		C.	Sin. $\tfrac{1}{2}$ P × ſin. $\overline{\tfrac{1}{2} P - AB}$: ſin. $\overline{\tfrac{1}{2} P - AC}$ × ſin. $\overline{\tfrac{1}{2} P - BC}$ :: Rad.^{2} : tang.^{2} $\tfrac{1}{2}$ C. prop. 29. Or the angle C may be found from the angle A, thus. Sin. $\overline{\tfrac{1}{2} P - AB}$: ſin. $\overline{\tfrac{1}{2} P - BC}$ :: tang. $\tfrac{1}{2}$ A : tang. $\tfrac{1}{2}$ C. 4. cor. 29.

Put

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	Given.	Sought.	Solution.
6.	A, B, C; the three angles.	The ſides;	Put half the ſum of the angles, $\frac{A + B + C}{2} = \frac{1}{2} S$. Then
		B C,	Cos. $\frac{1}{2} S \times$ cos. $\overline{\frac{1}{2} S - A}$: cos. $\overline{\frac{1}{2} S - B} \times$ cos. $\overline{\frac{1}{2} S - C}$ :: Rad.^{2} : cot.^{2} $\frac{1}{2}$ B C. prop. 30.
		A C,	Cos. $\frac{1}{2} S \times$ cos. $\overline{\frac{1}{2} S - B}$: cos. $\overline{\frac{1}{2} S - A} \times$ cos. $\overline{\frac{1}{2} S - C}$ :: Rad.^{2} : cot.^{2} $\frac{1}{2}$ A C. prop. 30. Or the ſide A C may be found from the ſide B C, thus. Cos. $\overline{\frac{1}{2} S - A}$: cos. $\overline{\frac{1}{2} S - B}$ :: tang. $\frac{1}{2}$ B C : tang. $\frac{1}{2}$ A C. 4. cor. 30.
		A B.	Cos. $\frac{1}{2} S \times$ cos. $\overline{\frac{1}{2} S - C}$: cos. $\overline{\frac{1}{2} S - A} \times$ cos. $\overline{\frac{1}{2} S - B}$ :: Rad.^{2} : cot.^{2} $\frac{1}{2}$ A B prop. 30. Or the ſide A B may be found from the ſide B C, thus. Cos. $\overline{\frac{1}{2} S - A}$: cos. $\overline{\frac{1}{2} S - C}$: : tang. $\frac{1}{2}$ B C : tang. $\frac{1}{2}$ A B. 4. cor. 30. But this caſe may alſo be ſolved, as the preceding, by means of a ſupplemental triangle.

APPENDIX.

OF A DIFFERENT SOLUTION OF SOME CASES OF SPHERICAL TRIANGLES.

LEMMA I.

THE ſum of the ſines of two unequal angles, which are not the ſupplements of one another, is to the difference of thoſe ſines, as the tangent of half the ſum of thoſe angles to the tangent of half their difference.

Figure **XVIII.**

Let, firſt, the angles A B C, B C A be together leſſer than two right angles, therefore A B C is a triangle. And becauſe the ſine of the angle B is to the ſine of the angle C, as C A to A B [a]; by compoſition and converſion, the ſum of the ſines of the angles B, C is to the difference of thoſe ſines, as the ſum of the ſides C A, A B to the difference of thoſe ſides; and therefore as the tangent of half the ſum of the angles B, C to the tangent of half their difference [b].

Secondly, let the angles A C D, A B E be together greater than two right angles. Becauſe the ſine of the angle A C D is the ſame with the ſine of the angle A C B [c], and the ſine of A B E the ſame with the ſine of A B C [c]; therefore the ſum of the ſines of the angles A C D, A B E is the ſame with the ſum of the ſines of the angles ABC, BCA; and the difference of the ſines of the firſt angles the ſame with the difference of the ſines of the laſt angles. And becauſe the angles A C D, A B E, A B C, B C A are together equal to four right angles: therefore half the ſum of the angles A C D, A B E, together with half the ſum of the angles A B C, B C A, is equal to two right angles: wherefore half the ſum of the angles A B C, B C A is the ſupplement of half the ſum of the angles A C D, A B E; and ſo the tangents of thoſe half-ſums are the ſame [c]. Alſo, becauſe the angles A C D, A C B are together equal to the angles A B E, A B C; theſe four angles are therefore equidifferent, as appears from Arithmetick: wherefore the difference of the angles A C D, A B E is equal to the difference of

[a] 12. 1.

[b] 13. 1.

[c] 6. cor. 9. def. 1.

[d] 4. cor. 9. def. 1.

the angles A B C, B C A; and half the former difference is equal to half the latter difference; and the tangents of thoſe half-differences are equal [d]. From what therefore has been demonſtrated of the angles A B C, B C A, it is apparent of the angles A C D, A B E alſo, that the ſum of their ſines is to the difference of thoſe ſines, as the tangent of half the ſum of thoſe angles, to the tangent of half their difference.

L E M M M A II.

The ſum of the coſines of two unequal angles, which are not the ſupplements of one another, is to the difference of thoſe coſines, as the cotangent of half the ſum of thoſe angles, to the tangent of half their difference, if the angles are of the ſame affection; and as the tangent of half their ſum to the cotangent of half their difference, if the angles are of different affection.

Figure XVIII.

[a] cor. lem. 1.
[b] El. 4. 1.
[c] El. 32. 1.
[d] 1. lem. 2.

Let, firſt, the angles C A F, F A B be of the ſame affection; and taken together, leſſer than two right angles. Draw B C perpendicular to A F, and biſect the angle B A C by the ſtrait line A G: make A H equal to A B; and join B H, meeting A F, A G in the points K, L. Therefore the angle B A L is half the ſum of the angles C A F, F A B; and the angle G A F, which is the exceſs of the greater angle C A F above half the ſum C A G, is half the difference of thoſe angles [a]. Alſo, B L is perpendicular to A G [b]; and the angle A B H is equal to B H A [b], and is half the ſum of the angles A B C and C [c]; and the angle G B L is half the difference of the angles A B C and C [a]. And becauſe the angle G A F is equal to the angle G B L [c]; half the difference of the angles C A F, F A B is equal to half the difference of the angles A B C and C. But the ſum of the ſines of the angles A B C and C is to the difference of thoſe ſines, as the tangent of half the ſum of thoſe angles to the tangent of half their difference [d]. And the ſine of the angle C is the coſine of the angle C A F, and the ſine of A B C the coſine of F A B; and half the ſum of the angles A B C and C, i. e. the angle A B L, is the complement of the angle B A L, or of half the ſum of the angles C A F, F A B: and therefore the tangent of half the ſum of the angles A B C and C, is the cotangent of half the ſum of the angles C A F, F A B: alſo the tangent of half the difference of the angles A B C and C is equal to the tangent of half the difference of the angles C A F, F A B[e]; becauſe thoſe half-differences are equal. Wherefore the ſum of the coſines of the angles C A F, F A B is to the difference of thoſe coſines, as the cotangent of half the ſum of thoſe angles, to the tangent of half their difference. And the ſame is on the angles B A M, M A C, which are of the ſame affection, and taken together are greater than two right angles; as may be demonſtrated juſt as the 2d. part of the preceding lemma.

Secondly, let the angles N A F, F A C be of different affection. Produce N A to B, and make the preceding conſtruction. Therefore the ſine of the angle A B F is the coſine of the angle N A F[c], and the ſine of C the coſine of C A F: and the angle A B H, which is half the ſum of the angles A B C and C[c], is half the angle N A C[c]; which is the difference of the angles N A F, F A C; and ſo the tangent of half the ſum of the angles A B C and C, is equal to the tangent of half the difference of the angles N A F, F A C: and the angle A K B, which is the exceſs of the greater angle N A F above half the difference A B K[c], is half the ſum of the angles N A F, F A C[a]; and the angle F B K, which is half the difference of the angles A B C and C, is the complement of the angle A K B[c]: wherefore the tangent of half the difference of the angles A B C and C, is the cotangent of half the ſum of the angles N A F, F A C. Therefore again from the preceding lemma, the ſum of the coſines of the angles N A F, F A C, is to the difference of thoſe coſines; as the tangent of half the difference of thoſe angles to the cotangent of half their ſum; or as the tangent of half their ſum to the cotangent of half their difference[f].

The ſame thing is manifeſt in that caſe alſo, when one of the angles is right: for then the difference of the angles being the ſupplement of their ſum[g], half their difference is the complement of half their ſum; wherefore the tangent of half the difference is the cotangent of half the ſum, and the tangent of half the ſum the cotangent of half the difference. In that caſe alſo the ſum of the coſines of the angles is equal to the difference of thoſe coſines[h].

[e] 4. cor. 9. def. 1.

[c] El. 32. 1.

[a] cor. lem. 2.

[f] cor. 1. 1.

[g] 3. def. 1.

[h] ſchol. 4. 1.

LEMMA III.

The ſum of the tangents of two unequal angles, neither of which is right, nor the ſupplement of the other, is to the difference of thoſe tangents; as the ſine of the ſum of thoſe angles to the ſine of their difference, if the angles are of the ſame affection; and as the ſine of the difference to the ſine of the ſum, if the angles are of different affection.

Figure XVIII.

Let, firſt, the angle C A F, F A B be of the ſame affection, and taken together leſſer than two right angles. Draw B C, perpendicular to A F. and make F O equal to F B, and join AO. Therefore A B is equal to A O, and the angle B A F to the angle F A O[a]. And in the triangle A B C, as CF to F B, ſo is the tangent of the angle C A F to the tangent of the angle F A B[b]: wherefore, by compoſition and converſion, as B C to C O, ſo is the ſum of the tangents of the angles C A F, F A B, to the difference of thoſe tangents. And becauſe as B C to B A, ſo is the ſine of the angle B A C to the ſine of the angle C[c]; and as B A, or A O, to C O, ſo is the ſine of the angle C to the ſine of the angle C A O[c]: ex aequali, as B C to C O, ſo is the ſine of the angle B A C to the ſine of the angle C A O[d]; i. e. ſo is the ſine of the ſum of the angles C A F, F A B to the ſine of their difference. Therefore alſo the ſum of the tangents of the angles C A F, F A B, is to the difference of thoſe tangents, as the ſine of the ſum of thoſe angles to the ſine of their difference. And the ſame is on the angles B A M, M A C, which are of the ſame affection, and taken together are greater than two right angles; as may be demonſtrated, juſt as the 2d. part of lem. 1.

Secondly, let the angles P A F, F A B be of different affection. Produce P A to C, and make the preceding conſtruction. From what therefore has been demonſtrated, the ſum of the tangents of the angles C A F, F A B, is to the difference of thoſe tangents, as the ſine of the angle B A C to the ſine of the angle C A O. But the tangent of the angle C A F is the ſame with

[a] El. 4. 1.

[b] cor. 12. 1.

[c] 12. 1.

[d] El. 22. 5.

with the tangent of the angle PAF[e]: and the ſine of the angle BAC the ſame with the ſine of the angle PAB[e], which is the difference of the angles PAF, FAB: alſo the ſine of the angle CAO is the ſame with the ſine of the angle PAO[e], which is the ſum of thoſe angles[a]. Therefore the ſum of the tangents of the angles PAF, FAB is to the difference of thoſe tangents, as the ſine of the difference of thoſe angles, to the ſine of their ſum.

Cor. The ſum of the cotangents is to their difference in the ſame ratio. For the ſum of the cotangents is to their difference, as the ſum of the tangents to their difference[f].

It is manifeſt alſo, that theſe lemmas and cor. may be uſed on the arches of the ſame, or of equal circles.

LEMMA. IV.

The ſines of half the ſum and of half the difference of two arches, or angles, are reciprocally proportional to half of the radius, and to the difference of the verſed ſines of thoſe arches, or angles.

Fig. XIX.

Let D be the center of the circle ACB; and BE, CF the ſines of its two arches AB, AC: wherefore AE, AF are the verſed ſines of thoſe arches, and EF is their difference. Biſect the arch BC in G: BG then is half the difference of the arches AB, AC; and AG, which is the exceſs of the greater AB above half the difference BG, is half the ſum of thoſe arches[a]. Draw BC, DG, and alſo GH the ſine of the arch AG; and through C draw CK parallel to AD.

Wherefore CK is equal to EF[b], and half BC is equal to the ſine of half the difference BG[c]. And becauſe the triangles DGH, BCK are equiangular, as GH, the ſine of half the ſum AG, is to half of the radius DG, ſo is CK, or EF, i. e. the difference of the verſed ſines, to half BC[d], or to the ſine of half the difference BG.

THEOREM

[e] 4. cor. 9. def. 1.

[a] El. 4. 1.

[f] cor. 1. 1.

[a] cor. lem. 1.

[b] El. 34. 1.

[c] 1. cor. 4. def. 1.

[d] El. 4. 6.

THEOREM III.

Figure XI. XII. In every ſpherical triangle ABC, as the ſine of half the ſum of two unequal ſides BA, AC, to the ſine of half their difference; ſo is the cotangent of half the contained angle BAC, to the tangent of half the difference of the angles at the baſe, C, B: And as the coſine of half the ſum of the ſides BA, AC, to the coſine of half their difference; ſo is the cotangent of half the contained angle BAC, to the tangent of half the ſum of the angles at the baſe, C, B.

From A draw an arch of a great circle, AD, perpendicular to the baſe BC. And firſt, let AD fall within the triangle ABC.

[a] 24. 2. 1ſt. Sin. BA : ſin. AC :: ſin. C : ſin. B[a]. wherefore Sin. BA + ſin. AC : ſin. BA − ſin. AC :: ſin. C + ſin. B : ſin. C − ſin. B; i. e.

[b] 1. lem. 2. Tang. $\frac{BA+AC}{2}$: tang. $\frac{BA-AC}{2}$[b] :: tang. $\frac{C+B}{2}$: tang. $\frac{C-B}{2}$[b], and alſo

[c] cor. 1. 1. Cot. $\frac{BA+AC}{2}$: cot. $\frac{BA-AC}{2}$ :: tang. $\frac{C-B.}{2}$ tang. $\frac{C+B}{2}$[c].

[d] 26. 2. 2dly. Sin. BAD : ſin. DAC :: cos. B : cos. C[d]. wherefore Sin. BAD + ſin. DAC : ſin. BAD − ſin. DAC :: cos. B + cos. C : cos. B − cos. C; i. e.

[e] 2 lem. 2. & 16. 2. Tang. $\frac{1}{2}$ A : tang. $\frac{BAD-DAC}{2}$[b] :: cot. $\frac{C+B}{2}$: tang. $\frac{C-B}{2}$[e]; whence

Tang. $\frac{C+B}{2}$: tang. $\frac{BAD-DAC}{2}$ :: cot. $\frac{1}{2}$ A : tang. $\frac{C-B}{2}$[c]. and alſo

Tang. $\frac{C-B}{2}$: tang. $\frac{BAD-DAC}{2}$ :: cot. $\frac{1}{2}$ A : tang. $\frac{C+B}{2}$.

[f] 28. 2. 3dly. Cot. BA : cot. AC :: cos. BAD : cos. DAC[f]. wherefore

Cot.

$\text{Cot. } AC + \text{cot. } BA : \text{cot. } AC - \text{cot. } BA :: \text{cos. } DAC + \text{cos. } BAD :$ $\text{cos. } DAC - \text{cos. } BAD$; i. e.

$\text{Sin.} \overline{BA + AC} : \text{ſin.} \overline{BA - AC}$[g] $:: \text{cot.} \tfrac{1}{2} A : \text{tang.} \frac{BAD - DAC}{2}$[h]; and

[g] cor. 3. lem. 2.
[h] 2. lem. 2. & 17. 2.

$\tfrac{1}{2} \text{ſin.} \overline{BA + AC} : \tfrac{1}{2} \text{ſin.} \overline{BA - AC} :: \text{cot.} \tfrac{1}{2} A : \text{tang.} \frac{BAD - DAC}{2}.$

Therefore

$$\left.\begin{array}{l} \text{Tang.} \frac{BA+AC}{2} \\ \times \text{Tang.} \frac{C+B}{2} \\ \times \tfrac{1}{2} \text{Sin.} \overline{BA+AC} \end{array}\right\} : \left\{\begin{array}{l} \text{Tang.} \frac{BA-AC}{2} \\ \times \text{Tang.} \frac{BAD-DAC}{2} \\ \times \tfrac{1}{2} \text{Sin.} \overline{BA-AC} \end{array}\right\} :: \left\{\begin{array}{l} \text{Tang.} \frac{C+B}{2} \\ \times \text{cot.} \tfrac{1}{2} A \\ \times \text{cot.} \tfrac{1}{2} A \end{array}\right\} : \left\{\begin{array}{l} \text{Tang.} \frac{C-B}{2} \\ \times \text{Tang.} \frac{C-B}{2} \\ \times \text{Tang.} \frac{BAD-DAC}{2} \end{array}\right.$$

[k]. And

[k] El. D. 11.

$$\left.\begin{array}{l} \text{Tang.} \frac{BA+AC}{2} \\ \times \tfrac{1}{2} \text{Sin.} \overline{BA+AC} \end{array}\right\} : \left\{\begin{array}{l} \text{tang.} \frac{BA-AC}{2} \\ \times \tfrac{1}{2} \text{ſin.} \overline{BA-AC} \end{array}\right\} :: \text{cot.}^2 \tfrac{1}{2} A : \text{tang.}^2 \frac{C-B}{2}$$

[l]; i. e.

[l] El. 32. 11.

$\text{Sin.}^2 \frac{BA + AC}{2} : \text{ſin.}^2 \frac{BA - AC}{2}$[m] $:: \text{cot.}^2 \tfrac{1}{2} A : \text{tang.}^2 \frac{C - B}{2}$; whence

[m] 2. 1.

$\text{Sin.} \frac{BA + AC}{2} : \text{ſin.} \frac{BA - AC}{2} :: \text{cot.} \tfrac{1}{2} A : \text{tang.} \frac{C - B}{2}$[n].

[n] El. 22. 6.

$$\text{Again.} \left.\begin{array}{l} \text{Cot.} \frac{BA+AC}{2} \\ \times \text{Tang.} \frac{C-B}{2} \\ \times \tfrac{1}{2} \text{ſin.} \overline{BA+AC} \end{array}\right\} : \left\{\begin{array}{l} \text{cot.} \frac{BA-AC}{2} \\ \times \text{tang.} \frac{BAD-DAC}{2} \\ \times \tfrac{1}{2} \text{ſin.} \overline{BA-AC} \end{array}\right\} :: \left\{\begin{array}{l} \text{tang.} \frac{C-B}{2} \\ \times \text{cot.} \tfrac{1}{2} A \\ \times \text{cot.} \tfrac{1}{2} A \end{array}\right\} : \left\{\begin{array}{l} \text{tang.} \frac{C+B}{2} \\ \times \text{tang.} \frac{C+B}{2} \\ \times \text{tang.} \frac{BAD-DAC}{2} \end{array}\right.$$

[k]. And

$$\left.\begin{array}{l} \text{Cot.} \frac{BA+AC}{2} \\ \times \tfrac{1}{2} \text{ſin.} \overline{BA+AC} \end{array}\right\} : \left\{\begin{array}{l} \text{cot.} \frac{BA-AC}{2} \\ \times \tfrac{1}{2} \text{ſin.} \overline{BA-AC} \end{array}\right\} :: \text{cot.}^2 \tfrac{1}{2} A : \text{tang.}^2 \frac{C+B}{2}$$

[l]; i. e.

$\text{Cos.}^2 \frac{BA+AC}{2} : \text{cos}^2. \frac{BA-AC}{2}$[m] $:: \text{cot.}^2 \tfrac{1}{2} A : \text{tang.}^2 \frac{C + B}{2}$; and therefore alſo

$\text{Cos.} \frac{BA + AC}{2} : \text{Cos.} \frac{BA - AC}{2} :: \text{cot.} \tfrac{1}{2} A : \text{tang.} \frac{C + B}{2}$[n].

Secondly. The ſame analogies are had, when the perpendicular falls without the triangle ABC, in which caſe $\frac{BAD - DAC}{2} = \tfrac{1}{2} A$.

Thirdly. When one of the angles, as C, is right; then

[o] 24. 2. & 1 lem. 2.

1ſt. $\text{Tang.}\ \frac{BA+AC}{2} : \text{tang.}\ \frac{BA-AC}{2} :: \text{tang.}\ \frac{C+B}{2} : \text{tang.}\ \frac{C-B}{2}$[o]; and

[c] cor. 1. 1.

$$\text{cot.}\ \frac{BA+AC}{2} : \text{cot.}\ \frac{BA-AC}{2} :: \text{tang.}\ \frac{C-B}{2} : \text{tang.}\ \frac{C+B}{2}\text{[c]}.$$

2dly. Becauſe the angle C is right, $\frac{C-B}{2}$ is the complement of $\frac{C+B}{2}$; and $\text{cot.}\ \frac{C+B}{2} = \text{tang.}\ \frac{C-B}{2}$. wherefore

$\text{Tang.}\ \frac{1}{2}A : \text{cot.}\ \frac{C+B}{2} :: \text{tang.}\ \frac{1}{2}A : \text{tang.}\ \frac{C-B}{2}$. whence

$\text{Tang.}\ \frac{C+B}{2} : \text{tang.}\ \frac{1}{2}A :: \text{cot.}\ \frac{1}{2}A : \text{tang.}\ \frac{C-B}{2}$[c]; and alſo

$$\text{Tang.}\ \frac{C-B}{2} : \text{tang.}\ \frac{1}{2}A :: \text{cot.}\ \frac{1}{2}A : \text{tang.}\ \frac{C+B}{2}.$$

[p] 21. 2.

3dly. Cot. B A : cot. A C :: cos. A : Rad[p]. wherefore Cot. A C + B A : cot. A C − cot. BA :: Rad. + cos. A : Rad − cos. A; i. e.

[q] cor. & lem. 2.
[r] 3. cor. 9. def. 1; cor. 2. 1. & 2. cor. 15. 2.

$\text{Sin.}\ \overline{BA+AC} : \frac{1}{2}\,\text{ſin.}\ \overline{BA-AC}$[q] $:: \text{cot.}\ \frac{1}{2}A : \text{tang.}\ \frac{1}{2}A$[r]; and $\frac{1}{2}\,\text{ſin.}\ \overline{BA+AC} : \frac{1}{2}\,\text{ſin.}\ \overline{BA-AC} :: \text{cot.}\ \frac{1}{2}A : \text{tang.}\ \frac{1}{2}A$. Therefore in this caſe alſo the ſame analogies, as before, will come out.

THEOREM IV.

In every ſpherical triangle, as the ſine of half the ſum of two unequal angles, to the ſine of half their difference; ſo is the tangent of half the ſide, adjacent to thoſe two angles, to the tangent of half the difference of the other two ſides: And as the coſine of half the ſum of thoſe two angles, to the coſine of half their difference; ſo is the tangent of half the

ſide, adjacent to thoſe two angles, to the tangent of half the ſum of the other two ſides.

This appears from the preceding theorem and a ſupplemental triangle.

Figure XI. XII.

Cor. 1. In a ſpherical triangle A B C, two ſides B A, A C, and the contained angle A, being given, the reſt is given. For Sin. $\frac{BA+AC}{2}$: ſin. $\frac{BA-AC}{2}$ :: cot. $\frac{1}{2}$ A : tang. $\frac{C-B}{2}$ [a]; which is leſſer than a right angle[b]. And cos. $\frac{BA+AC}{2}$: cos. $\frac{BA-AC}{2}$ :: cot. $\frac{1}{2}$ A : tang. $\frac{C+B}{2}$ [a], which is of the ſame affection with half the ſum of the ſides B A, A C [c]. therefore each of the angles C, B is given[d]. Alſo, Sin. $\frac{C-B}{2}$: ſin. $\frac{C+B}{2}$ :: tang. $\frac{BA-AC}{2}$: tang. $\frac{1}{2}$ B C[e]; or Cos. $\frac{C-B}{2}$: cos. $\frac{C+B}{2}$ :: tang. $\frac{BA+AC}{2}$: tang. $\frac{1}{2}$ B C[e]. But B C may alſo be found without the previous ſeeking of the angles C, B. for Rad.^{2} : ſin. B A × ſin. A C :: (ſin.^{2} $\frac{1}{2}$ A : ſin. $\overline{\tfrac{1}{2}BC+\frac{BA-AC}{2}}$ × ſin. $\overline{\tfrac{1}{2}BC-\frac{BA-AC}{2}}$[f] :: $\frac{1}{2}$ Rad. × vers. ſin. A[g] : $\frac{1}{2}$ Rad. × $\overline{\text{vers. ſin. } BC - \text{vers. ſin. } \overline{BA-AC}}$[h] : :) vers. ſin. A : vers. ſin. B C — vers. ſin. $\overline{BA-AC}$[k]; if therefore vers. ſin. $\overline{BA-AC}$ be added, there is had vers. ſin. B C. Or Rad.^{2} : ſin. A B × ſin. A C :: vers. ſin. ſuppl. A : vers. ſin. $\overline{BA+AC}$ — vers. ſin. B C; which being taken from vers. ſin. B A + A C, leaves vers. ſin. B C.

Cor. 2. In a ſpherical triangle A B C, two angles C, B, and the ſide adjacent to them, B C, being given, the reſt is given. For ſin. $\frac{C+B}{2}$: ſin. $\frac{C-B}{2}$ :: tang. $\frac{1}{2}$ B C : tang. $\frac{BA-AC}{2}$ [e], which is leſſer than a quadrant[l]. And Cos. $\frac{C+B}{2}$: Cos. $\frac{C-B}{2}$ :: tang. $\frac{1}{2}$ B C : tang.

[a] 3. th. 2.

[b] 2. cor. 6. 2

[c] 1. cor. 12. 2.

[d] lem. 1.

[e] 4. th. 2.

[f] 2. cor. 29. 2.

[g] 2. 1.

[h] 4. lem. 2.

[k] El. 1. 6.

[l] 9. def. 2.

$\frac{BA+AC}{2}$[e], which is of the ſame affection with the half ſum of the angles C, B[c]. therefore each of the ſides B A, A C is given[d]. Alſo, Sin. $\frac{BA+AC}{2}$: ſin. $\frac{BA-AC}{2}$ :: cot. $\frac{C-B}{2}$: tang. $\frac{1}{2}$ A[m]; Or Cos. $\frac{BA+AC}{2}$: cos. $\frac{BA-AC}{2}$ :: cot. $\frac{C+B}{2}$: tang. $\frac{1}{2}$ A[m]. But the angle A may alſo be found without the previous ſeeking of the ſides B A, A C. for Rad.^{2} : ſin. B × ſin. C :: vers. ſin. ſuppl. B C : vers. ſin. ſuppl. A — vers. ſin. $\overline{C-B}$; if therefore vers. ſin. $\overline{C-B}$ be added, there is had vers. ſin. ſuppl. A. Or Rad.^{2} : ſin. B × ſin. C :: vers. ſin. B C : vers. ſin. $\overline{C+B}$ — vers. ſin. ſuppl. A; which being taken from vers. ſin. $\overline{C+B}$, leaves vers. ſin. ſuppl. A. And the laſt two analogies may appear from the laſt two analogies of the preceding cor. and a ſupplemental triangle.

Cor. 3. In a ſpherical triangle A B C, two ſides B A, A C, and one of the oppoſite angles, B, being given, the reſt is given. For the angle C is given by prop. 24. Then Sin. $\frac{C-B}{2}$: ſin. $\frac{C+B}{2}$ :: tang. $\frac{BA-AC}{2}$: tang. $\frac{1}{2}$ B C[e]; Or Cos. $\frac{C-B}{2}$: cos. $\frac{C+B}{2}$ :: tang. $\frac{BA+AC}{2}$: tang. $\frac{1}{2}$ B C[e]. Alſo, Sin. $\frac{BA+AC}{2}$: ſin. $\frac{BA-AC}{2}$ :: cot. $\frac{C-B}{2}$: tang. $\frac{1}{2}$ A;[m] Or Cos. $\frac{BA+AC}{2}$: cos. $\frac{BA-AC}{2}$ :: cot. $\frac{C+B}{2}$: tang. $\frac{1}{2}$ A[m].

Cor. 4. In a ſpherical triangle A B C, two angles B, C, and one of the oppoſite ſides, A B, being given, the reſt is given. For the ſide A C is given from prop. 24. Then ſin. $\frac{BA+AC}{2}$: ſin. $\frac{BA-AC}{2}$ :: cot. $\frac{C-B}{2}$: tang. $\frac{1}{2}$ A[m]; Or Cos. $\frac{BA+AC}{2}$: cos. $\frac{BA-AC}{2}$ :: cot. $\frac{C+B}{2}$: tang. $\frac{1}{2}$ A[m]. Alſo, Sin. $\frac{C-B}{2}$: ſin. $\frac{C+B}{2}$ :: tang. $\frac{BA-AC}{2}$: tang. $\frac{1}{2}$ B C[e]; Or Cos. $\frac{C-B}{2}$: cos. $\frac{C+B}{2}$ :: tang. $\frac{BA+AC}{2}$: tang. $\frac{1}{2}$ B C[e].

[e] 4. th. 2.

[c] 1. cor. 12. 2

[d] lem. 1.

[m] 3. th. 2. & cor. 1. 1.

Fig. XV.

Cor. 5. In a rightangled ſpherical triangle ABC, having the angle A right, Sin. $\overline{BC + BA}$: ſin. $\overline{BC - BA}$ :: Rad.^{2} : tang.^{2} $\frac{1}{2}$ B. For it has been demonſtrated in the preceding theor. caſe 3d. that Sin. $\overline{BC + BA}$: ſin $\overline{BC - BA}$ :: cot. $\frac{1}{2}$ B : tang. $\frac{1}{2}$ B ; but cot. $\frac{1}{2}$ B : tang. $\frac{1}{2}$ B :: Rad.^{2} : tang.^{2} $\frac{1}{2}$ B [n].

[n] cor. 1. 1. & El. 2. cor. 20. 6.

Cor. 6. In a quadrantal ſpherical triangle ABC, having the ſide BC a quadrant, Sin. $\overline{A + B}$: ſin. $\overline{A - B}$ :: Rad.^{2} : cot.^{2}. $\frac{1}{2}$ AB ; as appears from the preceding cor. and a ſupplemental triangle.

THEOREM V.

Figure XI. XII.

In every ſpherical triangle ABC, if from the vertex A be drawn an arch of a great circle, AD, perpendicular to the baſe BC; the tangents of half the ſum and of half the difference of the ſides BA, AC, are reciprocally proportional to the tangents of half the ſum and of half the difference of the ſegments of the baſe, BD, DC.

For Cos. BA : cos. AC :: cos. BD : cos. DC [a] ; wherefore Cos. BA + cos. AC : cos. BA − cos. AC :: cos. BD + cos. DC : cos. BD − cos. DC ; i. e.

$$\text{Cot.}\,\frac{BA+AC}{2} : \text{tang.}\,\frac{BA-AC}{2} :: \text{cot.}\,\frac{BD+DC}{2} : \text{tang.}\,\frac{BD-DC}{2}\,^{b}\text{; or}$$

$$\text{Tang.}\,\frac{BA+AC}{2} : \text{cot.}\,\frac{BA-AC}{2} :: \text{tang.}\,\frac{BD+DC}{2} : \text{cot.}\,\frac{BD-DC}{2}\,^{b}.$$

Therefore

$$\text{Tang.}\,\frac{BD+DC}{2} : \text{tang.}\,\frac{BA+AC}{2} :: \text{tang.}\,\frac{BA-AC}{2} : \text{tang.}\,\frac{BD-DC}{2}\,^{c}.$$

[a] 27. 2.

[b] 2. lem. 2. & 17. 2.

[c] cor. 1. 1.

Cor. 1. In like manner from prop. 20. it may be demonſtrated that in a rightangled ſpherical triangle, the tangent of half of either of the ſides about the right angle, is a mean proportional between the tangents of half the ſum and of half the difference of the hypotenuſe and the other ſides.

Cor. 2. In a quadrantal ſpherical triangle, the cotangent of half of either of the angles adjacent to the quadrant, is a mean proportional between the tangents of half the ſum and of half the difference of the other two angles; as appears from the preceding cor. and a ſupplemental triangle.

Cor. 3. In a ſpherical triangle A B C, the three ſides being given, the angles alſo are given. For the perpendicular A D being drawn, it will be Tang. $\frac{1}{2}$ B C : tang. $\frac{BA + AC}{2}$ :: tang. $\frac{BA - AC}{2}$: tang. of ſome arch[d], leſſer than a quadrant[e]. And if this arch is equal to $\frac{1}{2}$ B C, the perpendicular coincides with one of the two ſides B A, A C; which ſide may be determined from prop. 11. if the ſides are of the ſame affection; and from prop. 12. if they are of different affection; and the triangle A B C will be rightangled. But if the fourth proportional arch be leſſer, or greater than $\frac{1}{2}$ B C, the perpendicular falls within, or without, the triangle A B C, and the ſegments of the baſe are found by lem. 1. And if the two ſides B A, A C are leſſer than a ſemicircumference; the greater ſegment of the baſe is adjacent to the greater of the two ſides B A, A C, as in plane triangles: but if thoſe two ſides are greater than a ſemicircumference, it will be adjacent to the leſſer of them; as may be demonſtrated from prop. 3. Laſtly, the angles B, C, or the ſupplement of one of them, as alſo the angle A, may be found by means of the two rightangled triangles A B D, A C D.

[d] 5. th. 2.

[e] 9. def. 2.

THEOREM VI.

Figure XI. XII.

The ſame things being ſuppoſed, the tangents of half the ſum and of half the difference of the angles at the baſe, B, C, are reciprocally proportional to the cotangent of half the angle at the vertex, B A C, and to the tangent of half the difference, or of half the ſum, of its ſegments B A D, D A C; of half the difference, when the perpendicular falls within the triangle; and of half the ſum, when it falls without.

For Cos. B : cos. C :: ſin. B A D : ſin. D A C[a]; wherefore Cos. B + cos. C : cos. B − cos. C :: ſin. B A D + ſin. D A C : ſin. B A D − ſin. D A C; i. e.

$$\text{Cot.}\,\frac{B+C}{2} : \text{tang.}\,\frac{B-C}{2} :: \text{tang.}\,\frac{BAD+DAC}{2}\ (=\text{tang.}\,\tfrac{1}{2}A) : \text{tang.}\,\frac{BAD-DAC}{2}$$

[b]; or

$$\text{Tang.}\,\frac{B+C}{2} : \text{cot.}\,\frac{B-C}{2} :: \text{tang.}\,\frac{BAD+DAC}{2} : \text{tang.}\,\frac{BAD-DAC}{2}$$

(= tang. ½ A.)[b]. Therefore

$$\text{Cot.}\,\tfrac{1}{2}A : \text{tang.}\,\frac{B+C}{2} :: \text{tang.}\,\frac{B-C}{2} : \text{tang.}\,\frac{BAD-DAC}{2},$$

[c] when the perpendicular is within, and

$$\text{Cot.}\,\tfrac{1}{2}A : \text{tang.}\,\frac{B+C}{2} :: \text{tang.}\,\frac{B-C}{2} : \text{tang.}\,\frac{BAD+DAC}{2},$$

[c] when the perpendicular is without.

Cor. In a ſpherical triangle A B C, the three angles being given, the ſides alſo are given. For the perpendicular A D being drawn, it will be $\text{Cot.}\,\tfrac{1}{2}A : \text{tang.}\,\frac{B+C}{2} :: \text{tang.}\,\frac{B-C}{2}$: tang. of ſome angle[d], leſſer than a right angle[e]. This angle is half the difference of the ſegments of the angle at the vertex, B A C, when the angles B, C are of the ſame affection[f]; and half their ſum, when they are of different affection[f]. In the firſt caſe, ½ B A C is half the ſum of the two angles B A D, D A C; in the ſecond, half their difference, whence the angles themſelves will be found, by lem. 1. And if the two angles B, C are together leſſer than two right angles, the greater of the ſegments of the angle at the vertex will be with the leſſer of them in one and the ſame of the two rightangled triangles A B D, A C D, as in plane triangles; but if thoſe two angles are together greater than two right angles, it will be with the greater of them in the ſame rightangled triangle; as may appear from 3. cor. of the preceding theor. and prop. 14. and 2. cor. 19. Laſtly, the ſides C A, A B, B C will be found by means of the two rightangled triangles A B D, A C D.

THE END.

[a] 26. 2.

[b] 2. & 1. lem. 2. & cor. 16. 2.

[c] cor. 1. 1.

[d] 6. th. 2.

[e] 2. cor. 6. 2.

[f] cor. 16. 2.

Cor. 2. In a quadrantal ſpherical triangle, the cotangent of half of either of the angles adjacent to the quadrant, is a mean proportional between the tangents of half the ſum and of half the difference of the other two angles; as appears from the preceding cor. and a ſupplemental triangle.

Cor. 3. In a ſpherical triangle A B C, the three ſides being given, the angles alſo are given. For the perpendicular A D being drawn, it will be Tang. $\frac{1}{2}$ B C : tang. $\frac{BA + AC}{2}$ :: tang. $\frac{BA - AC}{2}$: tang. of ſome arch[d], leſſer than a quadrant[e]. And if this arch is equal to $\frac{1}{2}$ B C, the perpendicular coincides with one of the two ſides B A, A C; which ſide may be determined from prop. 11. if the ſides are of the ſame affection; and from prop. 12. if they are of different affection; and the triangle A B C will be rightangled. But if the fourth proportional arch be leſſer, or greater than $\frac{1}{2}$ B C, the perpendicular falls within, or without, the triangle A B C, and the ſegments of the baſe are found by lem. 1. And if the two ſides B A, A C are leſſer than a ſemicircumference; the greater ſegment of the baſe is adjacent to the greater of the two ſides B A, A C, as in plane triangles: but if thoſe two ſides are greater than a ſemicircumference, it will be adjacent to the leſſer of them; as may be demonſtrated from prop. 3. Laſtly, the angles B, C, or the ſupplement of one of them, as alſo the angle A, may be ſound by means of the two rightangled triangles A B D, A C D.

THEOREM VI.

Figure XI. XII.

The ſame things being ſuppoſed, the tangents of half the ſum and of half the difference of the angles at the baſe, B, C, are reciprocally proportional to the cotangent of half the angle at the vertex, B A C, and to the tangent of half the difference, or of half the ſum, of its ſegments B A D, D A C; of half the difference, when the perpendicular falls within the triangle; and of half the ſum, when it falls without.

---

[d] 5. th. 2.

[e] 9. def. 2.

For Cos. B : cos. C :: ſin. BAD : ſin. DAC[a]; wherefore Cos. B + cos. C : cos. B − cos. C :: ſin. BAD + ſin. DAC : ſin. BAD − ſin. DAC; i. e.

Cot. $\frac{B+C}{2}$: tang. $\frac{B-C}{2}$ :: tang. $\frac{BAD+DAC}{2}$ (= tang. $\frac{1}{2}$ A) : tang. $\frac{BAD-DAC}{2}$[b]; or

Tang. $\frac{B+C}{2}$: cot. $\frac{B-C}{2}$ :: tang. $\frac{BAD+DAC}{2}$: tang. $\frac{BAD-DAC}{2}$ (= tang. $\frac{1}{2}$ A.)[b]. Therefore

Cot. $\frac{1}{2}$ A : tang. $\frac{B+C}{2}$[c] :: tang. $\frac{B-C}{2}$: tang. $\frac{BAD-DAC}{2}$, when the perpendicular is within, and

Cot. $\frac{1}{2}$ A : tang. $\frac{B+C}{2}$[c] :: tang. $\frac{B-C}{2}$: tang. $\frac{BAD+DAC}{2}$, when the perpendicular is without.

Cor. In a ſpherical triangle ABC, the three angles being given, the ſides alſo are given. For the perpendicular AD being drawn, it will be Cot. $\frac{1}{2}$ A : tang. $\frac{B+C}{2}$ :: tang. $\frac{B-C}{2}$: tang. of ſome angle[d], leſſer than a right angle[e]. This angle is half the difference of the ſegments of the angle at the vertex, BAC, when the angles B, C are of the ſame affection[f]; and half their ſum, when they are of different affection[f]. In the firſt caſe, $\frac{1}{2}$ BAC is half the ſum of the two angles BAD, DAC; in the ſecond, half their difference, whence the angles themſelves will be found, by lem. 1. And if the two angles B, C are together leſſer than two right angles, the greater of the ſegments of the angle at the vertex will be with the leſſer of them in one and the ſame of the two rightangled triangles ABD, ACD, as in plane triangles; but if thoſe two angles are together greater than two right angles, it will be with the greater of them in the ſame rightangled triangle; as may appear from 3. cor. of the preceding theor. and prop. 14. and 2. cor. 19. Laſtly, the ſides CA, AB, BC will be found by means of the two rightangled triangles ABD, ACD.

THE END.

[a] 26. 2.

[b] 2. & 1. lem. 2. & cor. 16. 2.

[c] cor. 1. 1.

[d] 6. th. 2.

[e] 2. cor. 6. 2.

[f] cor. 16. 2.

K
C
R
A
H
B
K

PLANE TRIGONOMETRY.

Plate I.

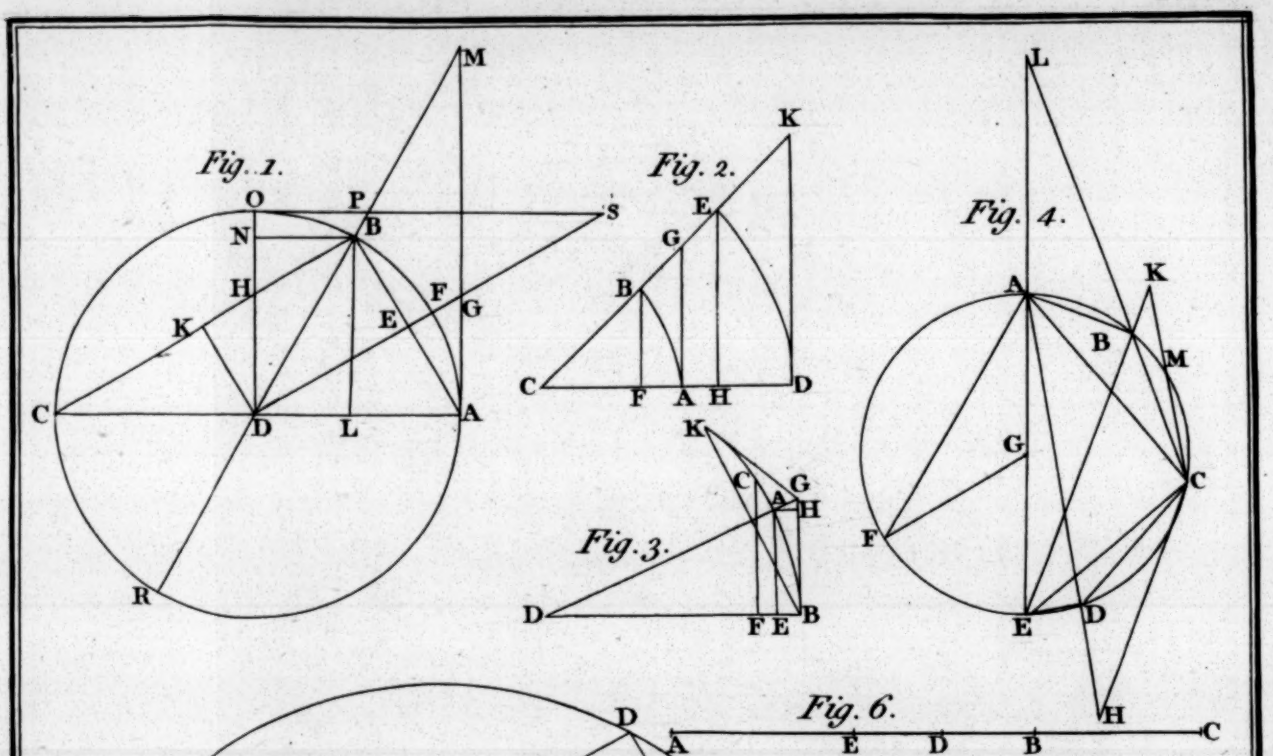


Fig.

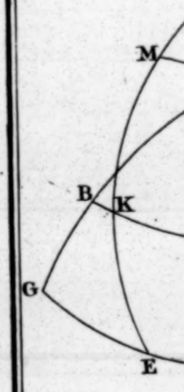


Fig.13

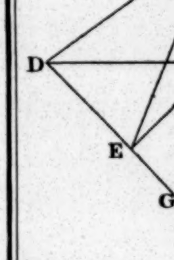
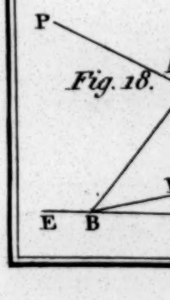


Fig.18.



SPHERICAL TRIGONOMETRY.

Plate II.